New families of distributions fitting L-moments for modelling financial data

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Background

Financial markets are usually represented by gaussian random variables.

Theories dealing with deviations from gaussian regimes are designed to fit certain characteristics of markets: fat tails, correlation switching, stochastic volatility, etc.

The two basic observables used to measure univariate gaussian properties are the mean and the variance, the first and second moment.

Because of this, attempts to capture non-gaussian behavior usually concentrate on higher moments:

- The third, which measures skewness
- The fourth, which measures spread.

The basic problem with this approach is that the higher moments are very unstable and highly dependent on extreme events.

Quantile Functions

We review some elementary properties and definitions. Let $F(x)$ be the distribution function of a random variable X. We shall use the symbol M_p to denote the moment functional of order p given by

$$M_p(F) = E(X^p) = \int_{-\infty}^{+\infty} x^p dF(x)$$

We shall denote by $Q(u)$ the associated quantile function of the distribution F, defined by

$$Q(u) \; = \; \inf \{x : \; F(x) \; \geq \; u\}$$

In what follows we shall assume that both F and Q are continuously differentiable.

L-moments as order statistics

Let $q(u) = Q'(u)$, be the density quantile function of F. The random variables $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ shall denote the order statistics associated to the distribution F. Think of it as the ordered values of a single sample of size n.

The L-moments of a population, $L_r(F)$, $r = 1, 2, \ldots$, were originally defined by Hosking(1990), as linear combinations of the expectations of $X_{i:n}$

$$L_r(F) = \frac{1}{r} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} E(X_{r-j:r}) , \qquad r = 1, 2, \ldots$$

Descriptive measures based on L moments were introduced by Hosking (1990) together with a clarifying discussion of their intuitive meanings.

Characteristics of the distribution

- The first L-moment $L_1(F)$ is the mean of F a measure of location.
- $L_2(F)$ is a scale measure, being half the value of Gini's mean difference.
- The L-moment ratios

$$\tau_r(F) = \frac{L_r(F)}{L_2(F)}$$

are scale free measures of the shape of a distribution. In particular, $\tau_3(F)$ and $\tau_4(F)$ are measures of skewness and kurtosis respectively.

L-moments as Fourier coefficients

From a technical viewpoint, it is often more convenient to adopt an equivalent definition of L-moments as the Fourier coefficients of the quantile function $Q(u)$ in terms of orthogonal polynomials on the interval [0,1] given by Hosking (1990),

$$L_r(F) \ = \ \int_0^1 Q(u)P_{r-1}(u)du, \qquad r = 1, 2, \ldots$$

where $P_r(u)$ are the shifted Legendre orthogonal polynomials on the interval $(0, 1)$. By definition

$$P_{r-1}(u) \ = \ \sum_{j=0}^{r-1} p_{r,j}\, u^j$$

where

$$p_{r,j} = \frac{(-1)^{r-j}(r+j)!}{(j!)^2(r-j)!}$$

Distribution reconstruction

The problem with reconstructing distributions from L-moments is their non-linearity.

We address this issue with a representation theorem which will ultimately lead to linear relationships which can be addressed with L-moments.

Theorem : *A generic quantile function has a measure λ such that*

$$Q(u) = \int_0^1 \chi_a(u)\, d\lambda(a)$$

where χ_a is a spectral function given by

$$\chi_a(u) = \begin{cases} b(a) \cdot 1_{[a,1]}(u) & a \geq \frac{1}{2} \\ b(a) \cdot \cdot 1_{[0,a]}(u) & a < \frac{1}{2} \end{cases}$$

and b is a GROWTH function:

- *b is increasing and continuous on the unit interval except at 1/2.*
- $b(1/2-) = -b(1/2+)$.
- $b < 0$ *when* $a < \frac{1}{2}$, *and* $b > 0$ *when* $a > \frac{1}{2}$.
- $b(0) = -\infty = -b(1)$.

Distribution reconstruction

Suppose $Q(u) \in \Psi_0$ and $\int_0^1 Q(u)du < \infty$. Let λ be the spectral measure associated to $Q(u)$. Then the following equations hold:

$$\int_0^1 P_{r-1}(u)Q(u)du = \int_0^1 P^*_{r-1}(a)\, d\lambda(a) \quad r = 1, \ldots, k,$$

where

$$P^*_{r-1}(a) = \begin{cases} \sum_{i=0}^{r-1} p_{i,r-1} \frac{(1-a^{i+1})b(a)}{(i+1)} & \text{if } a > u_0 \\ \sum_{i=0}^{r-1} p_{i,r-1} \frac{a^{i+1}b(a)}{(i+1)} & \text{if } a < u_0 \end{cases}$$

Entropy calibration

The method to reconstruct the distribution from sample observations will use the maximum entropy method to regularize the results.

Hence, we will seek a measure $x(a)$ that minimizes

$$\int_0^1 x(a) \log(x(a)) \, da$$

subject to the constraints derived from the L-moments:

$$\int_0^1 P^*_{r-1}(a) \, x(a) \, da = l_r, \qquad r = 2, \ldots, k.$$

By Borwein-Lewis, the solution is given by

$$x(a) = \exp\left(\sum_{i=1}^{k} \lambda_i \, P^*_{r-1}(a) \, da \right),$$

for certain λ_i.

Addressing unimodal distributions

The flexibility of the calibration method allows us to adapt the techniques to guarantee that the resulting distribution is, for instance, unimodal. This can be obtained by noting that the extremal elements in the set of unimodal distributions can be easily characterized by growth functions and spectral functions defined by

$$\chi_a(u) = \begin{cases} \frac{b(a)}{1-a}(x-a)\,1_{[a,1]} & \text{if } u_0 \geq a \\ \frac{b(a)}{a}(a-x)1_{[0,a]} & \text{if } a < u_0 \end{cases}$$

The rest of the process is as before.