Defaultable forward contracts. Pricing and Modelling.

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Introduction

- Future contracts cannot default but forward contracts can; We need to bring default events into the pricing of these OTC transactions.

- Common practice, forward prices in a forward contract are shocked "arbitrarily" to take defaults into account.

- The relation between default events and commodity prices can be represented as a new Term Structure called "defaultable forward prices" ($\overline{Fo}$).

- The best known example of derivatives in commodities, where default events are considered, are the vulnerable options.

What is a forward contract, FC.

A agrees to buy a designated good on a specified future date, T, at the strike price K, prevailing at the time the contract is initiated, t. No money changes hands initially or during the lifetime of the contract, only at T.

$$FC(t,T) = E_t^Q \left[e^{-\int_t^T r(u)du} (S(T) - K) \right] \tag{1}$$

Definition 1. ***What is a forward price,*** $Fo(t,T)$. *It is the value of the strike price such that forward contracts have zero value when they are initiated ($FC(0,T) = 0$).*

What is a Defaultable forward contract, DFC.

Simplest Case: **A** agrees to buy a good at price K at time T from counterparty **B**; if **B** does not default, **A** receives $S(T) - K$ from **B**; in case of a default from **B**, **A** receives nothing.

$$DFC(t,T) = E_t^Q \left[\exp\left\{ - \int_t^T r(s)ds \right\} \cdot \{(S(T) - K) \cdot (1 - N(T))\} \right] \quad (2)$$

Definition 2. ***Defaultable forward price,*** $\overline{Fo}(t,T)$. *It is the strike price such that a DFC has zero value when it is initiated* ($DFC(0,T) = 0$)*.*

Other Cases:

If **B** defaults and $K > S(T)$ then **A** pays $K - S(T)$ (real DFC).

Both **A** and **B** may default (two sided DFC).

Examples where this structure is needed: Crash of 1998. Billions of dollars are moved annually on OTC transactions involving defaultable parties.

Mathematical Problem: We wish to price DFC as well as describe the newly implied term structure "defaultable forward prices" under a risk neutral "Q" measure.

Credit Derivatives

Reduced form framework. Notation

- $B(t,T)$, $\overline{B}(t,T))$, bond and defaultable bond prices.
- $S(t)$, Commodity spot price, $Fo(t,T)$ forward price, $F(t,T)$ future prices.
- $N(t)$ - Cox Process with stochastic intensity $\lambda(t)$.
- Forward rate, $f(t,T) = -\dfrac{\partial}{\partial T}\ln(B(t,T))$.
- Defaultable forward rate, $\overline{f}(t,T) = -\dfrac{\partial}{\partial T}\ln(\overline{B}(t,T))$.
- Forward convenience yield, $\varepsilon(t,T) = -\dfrac{\partial}{\partial T}(\frac{Fo(t,T)}{S(t)B(t,T)})$.

The processes (drift) for S, f, $\overline{f}$ and $\varepsilon(t,T)$ are known under the Q-measure:

- Heath-Jarrow-Morton 1991 provides the drift of $f(t,T)$ under general conditions Most interest rate models are particular cases of this Framework.
- Schwartz 1997 generalizes HJM for commodities by finding the drift of $\varepsilon(t,T)$. The drift of $S(t)$ is known from Black-Scholes.
- Schonbucher 2001 generalizes HJM, describing the drift of $\overline{f}(t,T)$in a risk neutral world.

These results will be used to describe the drift of the defaultable forward price $\overline{Fo}(t,T)$ in the risk neutral world.

Inspired by the frameworks of:

HJM 1991 for Bonds $B(t,T) = e^{\int_t^T f(t,s)ds}$

Schwartz 1997 for forward prices $Fo(t,T) = S_t \cdot e^{\int_t^T (f(t,s)-\delta(t,s))ds}$

Schonbucher for defaultable Bonds $\overline{B}(t,T) = e^{\int_t^T \overline{f}(t,s)ds}$

We propose the following model for $\overline{Fo}(t,T)$. The idea is to decompose the term structure into a suitable set of factors:

$$\overline{Fo}(t,T) = S_t \cdot \exp\left\{ \int_t^T \left(\overline{f}(t,s) - \overline{\varepsilon}(t,s) \right) ds \right\}, \tag{3}$$

The term $\overline{\varepsilon}(t,s)$ is called defaultable instantaneous convenience yields.

Theorem 1. *The drift of* $\overline{Fo}(t,T)$, $\overline{\mu}^Q_{Fo}$ *and the drift of* $\overline{\varepsilon}(t,T)$, $\overline{\mu}^Q_{\varepsilon}$, *in the absence of arbitrage, are:*

$$\overline{\mu}^Q_{Fo}(t,T) = F(t,T) \times \frac{\partial G_0(t,T,\sigma_S,\sigma_\lambda,\sigma_\varepsilon,\mu^Q_\varepsilon,\mu^Q_f,\overline{\mu}^Q_f)}{\partial t} \tag{4}$$

$$\begin{aligned}\overline{\mu}^Q_\varepsilon(t,T) &= -\mu^Q_\varepsilon(t,T) + \mu^Q_f(t,T) - \overline{\mu}^Q_f(t,T) \\ &\quad + \frac{\partial^2 \left\{G_1(t,T,\sigma_S,\sigma_\lambda,\sigma_\varepsilon,\mu^Q_\varepsilon,\mu^Q_f,\overline{\mu}^Q_f)\right\}}{\partial T \partial t}\end{aligned} \tag{5}$$

respectively.

Other Credit Derivatives. This previous concept can be used as the backbone of a new breed of derivatives, for example:

1. **Options on DFC**: An standard option with maturity t on a defaultable forward contract starting at t maturity T, K stand by the strike price. $E_0^Q\left[\exp^{\left\{-\int_0^t r(s)ds\right\}} \cdot \left(\overline{Fo}(t,T) - K\right)^+\right]$
2. **Vulnerable options on DFC**: The issuer of the options may default before option's maturity day. There are two sources of default, the option seller, N_1, and the DFC underlying.
$E_0^Q\left[\exp^{\left\{-\int_0^t r(s)ds\right\}} \cdot \left(\overline{Fo}(t,T) - K\right)^+ \cdot (1 - N_1(T))\right]$

Structural Framework

Merton approach. Basic assumptions.

1. Defaults occurs at the maturity of the option, T, only if $V(T) < D$, where $V(t)$ stands for the option writer's assets (Merton approach 1974).
2. Zero recovery rate.

The Payoff of a defaultable forward contract in this context is:

$$E_t^Q \left[e^{\left\{ -\int_t^T r(s)ds \right\}} \cdot (S(T) - K) \cdot (1 - 1_{V(T)<D}) \right] \tag{6}$$

Results. We price one and two-sided DFC as well as vulnerable options on spot and futures prices.

Conclusions

- The notion of defaults, inherent on forward contracts, was added leading to an alternative derivative called defaultable forward contract.
- Defaultable forward prices were defined and modelled following standard frameworks.
- The idea was extended to other families of defaultable contracts as two sided DFC and real DFC.