The Essential Spectrum of Neumann Laplacians on some Bounded Singular Domains

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Abstract. In the present paper we consider Neumann Laplacians on singular domains of the type "rooms and passages" or "combs" and we show that, in typical situations, the essential spectrum can be determined from the geometric data. Moreover, given an arbitrary closed subset S of the non-negative reals, we construct domains $\Omega = \Omega(S)$ such that the essential spectrum of the Neumann Laplacians on Ω is just this set S.

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Introduction

Going back to Weyl's celebrated article on the asymptotics of Dirichlet regions in two dimensions, there is an enormous literature on Laplacians associated to regions of $\mathbf{R}^n$. Much of the literature is on the Dirichlet case which is easier, in part, because of compactness results. In this paper, we want to contribute to the study of the Neumann case, most particularly to identify the essential spectrum for Neumann Laplacians for some special regions.

Given an open region, Ω, in $\mathbf{R}^n$, we let $Q(-\Delta_N^\Omega)$ be the set of all functions in $L^2(\Omega)$ whose distributional gradients are in L^2 and we define $-\Delta_N^\Omega$ via the quadratic form relation

$$\langle \phi, -\Delta_N^\Omega \phi \rangle = \int |\nabla \phi(x)|^2 \; d^n x$$

It is a well-known result of Meyers and Serrin (cf. e. g. Adams [1], Gilbarg and Trudinger [10]), that the functions in $Q(-\Delta_N^\Omega)$ which are C^∞ in the interior are dense in $Q(-\Delta_N^\Omega)$. The closure of $C_0^\infty(\Omega)$ is the form domain of the Dirichlet Laplacian.

Dirichlet Laplacians of bounded regions have discrete spectrum since it is not hard to show their resolvents are compact. On the other hand, it has been known for many years that Neumann Laplacians of finite regions need not have purely discrete spectrum. Mind you, if the region is sufficiently regular, the Neumann Laplacian is compact — for example, if there is a piecewise smooth boundary. The following is an example of a region going back at least to Courant and Hilbert known as "rooms and passages".

To construct a typical rooms and passages–domain (as shown in Figure 1), take a sequence of *rooms* (= open rectangles R_k, contained in the unit ball of $\mathbf{R}^2$, $k \in \mathbf{N}$, R_k symmetric with respect to the x–axis, and such that $\overline{R}_k \cap \overline{R}_j = \emptyset, k \neq j$), which are joined together by *passages* (= rectangles P_k, $k \in \mathbf{N}$, P_k symmetric with respect to the x–axis) of height much smaller than the height of the adjoining rooms R_k and R_{k-1}.

Insert Figure 1 here.

If the passages are narrow enough, the Neumann Laplacian for this region has 0 in the essential spectrum. For, let ϕ_n be a function which is a large constant in the n–th room and which drops linearly to 0 between the room and the midpoint of the adjacent passages. Choose the constant so that ϕ_n has norm 1. Since they have disjoint supports, the ϕ_n are orthonormal. The size of $\|\nabla\phi_n\|$ is proportional to the width of the passages adjacent to box n and that can be made arbitrarily small.

One of the goals in this note is to actually show that for the rooms and passages example, the essential spectrum is exactly $\{0\}$, if the passages are narrow enough. Our main theorem is

Theorem 0.1. *Let S be any closed subset of $[0,\infty)$ and let n be given. Then there exists an open, connected subset Ω of the unit ball in $\mathbf{R}^n$ so that*

$$\sigma_{\text{ess}}(-\Delta_N^\Omega) = S \qquad \text{and} \qquad \sigma_{\text{ac}}(-\Delta_N^\Omega) = \emptyset$$

If S contains 0, we will be able to construct Ω as a modification of rooms and passages — essentially, we will add a partition accross each room with a hole in it. For general S, we will modify instead another class of regions known as "combs".

To construct combs, we attach a sequence of *teeth* (i.e., rectangles of bounded length and shrinking width) to a fixed square $Q \subset \mathbf{R}^2$. Here it is somewhat simpler to stack the teeth together instead of having empty space between them.

Insert Figure 2 here.

Basic to our entire strategy is that one can decouple into simpler regions. In the rooms and passages type regions, we'll decouple into separate rooms and passages2 in the combs, we'll decouple the teeth of the comb from the handle Q. In the rooms and passages, the barriers we put in will have Neumann boundary conditions on the room side and Dirichlet conditions on the passage side. What we'll show is that putting in such barriers on the infinity of room–passage joins will mean a compact perturbation of the resolvent so long as the passages are narrow enough (and a trace class perturbation on the level of square of the resolvent). Since σ_{ess} is left invariant under compact perturbations, we'll reduce the determination of the essential spectrum to that of decoupled regions. Since each individual region has discrete spectrum, the essential spectrum will be the set of limit points of spectra of the regions and that will be easy to compute.

The somewhat surprising element of our decoupling is that from the passage side, the boundary condition is Dirichlet. We call this the organ pipe lemma because it is a reflection of the known fact that closed and open organ pipes have opposite boundary conditions. The reason that eigenfunctions in the passages must vanish near the boundary of the passage is the following: Because the passages are so small, for them to matter, the wave function must live in the passage and not much in the rooms. If these functions weren't much smaller at the edge of the passage than in the middle, they'd "leak" out into the rooms. This idea is made precise in Section 1, where we investigate the behavior of eigenvalues, eigenfunctions and resolvents of the Neumann Laplacian on two domains Ω_1 and Ω_2 which are joined by a passage of width w, in the limit of w going to zero (Propositions 1.4 and 1.5). In Proposition 1.9, we deal in a similar manner with the situation where a family of small handles is attached to a fixed domain Ω_0. There has been previous discussion of the effect of narrow passages and handles, not unrelated to our organ pipe-lemma, see Jimbo [11] and Arrieta, Hale and Han [3].

In Section 2, we construct rooms and passages domains Ω by successively joining a sequence of rooms by narrow passages and obtain norm-resolvent convergence of certain approximating Laplacians H_n to $-\Delta_N^\Omega$. As in Simon and Spencer [16], the spectral results then follow from the fact that $(H_1+1)^{-2} - (-\Delta_N^\Omega+1)^{-2}$ is trace class. Upon replacing each room in the above construction by a small rectangle with a partition, each of these modified rooms will contribute (to the spectrum of H_1) an eigenvalue 0 plus another low-lying eigenvalue λ_k , while the remaining eigenvalues will be very large. By this construction, we can achieve to have

$$\sigma_{\text{ess}}(H_1) = \{0\} \cup \{\text{limit points of } \{\lambda_k\}\}.$$

(Domains similar to a union of finitely many rooms and passages have been used by Colin de Verdière [4] to specify a finite part of the Neumann spectrum.)

In an analogous manner, we analyze combs in Section 3, beginning with simple combs of the type described above and then proceeding to combs with small teeth D_k where each D_k has a partition (with "door") to make sure that each D_k contributes precisely one low-lying eigenvalue λ_k to the spectrum of the fully decoupled comparison operator, while the remaining eigenvalues of $-\Delta_N^{D_k}$ are very large. As a consequence, we find that the essential spectrum of the Neumann Laplacian is given as the limit set of the sequence $\{\lambda_k\}$. Since this sequence can be preassigned in the construction of examples, Theorem 0.1 follows.

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1. Organ–Pipes

In this section we analyze the Neumann Laplacian on domains consisting of two "rooms" which are joined by a narrow passage of width w, w small. It turns out that we have a natural Dirichlet boundary condition on the sides of the passage attached to the rooms (corresponding to the natural boundary condition for the pressure in an organ pipe at its open end). To be more precise, we will see that the resolvent of the Neumann Laplacian on the full domain is well approximated (in the operator norm) by the resolvent of a certain decoupled operator which has pure Neumann boundary conditions along the boundary of the two rooms and mixed Dirichlet and Neumann boundary conditions for the passage.

Here a word about the definition of Laplacians with Neumann or mixed boundary conditions is in order. For a general open domain $\Omega \subset \mathbf{R}^n$, the Neumann Laplacian is most naturally defined via quadratic forms, starting from the Sobolev space $\mathcal{H}^1(\Omega) = W^{1,2}(\Omega)$. This Sobolev space may be obtained as the completion of the function space

$$\left\{ f \in C^\infty(\Omega) \mid \|f\|_{\mathcal{H}^1(\Omega)} < \infty \right\}$$

under the norm $\|\cdot\|_{\mathcal{H}^1(\Omega)}$, where

$$\|f\|^2_{\mathcal{H}^1(\Omega)} = \int_\Omega |f|^2 + |\nabla f|^2$$

Then $-\Delta_N^\Omega$ is defined (as in Reed and Simon [15; Section XIII.15]) as the unique non-negative, self-adjoint operator whose domain $\mathcal{D}(-\Delta_N^\Omega)$ is contained and dense in $\mathcal{H}^1(\Omega)$, and which satisfies

$$\langle -\Delta_N^\Omega u, v \rangle = \langle \nabla u, \nabla v \rangle \qquad u \in \mathcal{D}(-\Delta_N^\Omega) \quad v \in \mathcal{H}^1(\Omega)$$

Similarly, Laplacians with mixed Neumann and Dirichlet boundary conditions can be defined in the following way: suppose $\Gamma = \bar{\Gamma} \subset \partial\Omega$ is given. Let

$$\mathcal{H}^1_\Gamma(\Omega) \subset \mathcal{H}^1(\Omega)$$

be the completion of

$$\left\{ f \in C^\infty(\Omega) \mid \|f\|_{\mathcal{H}^1(\Omega)} < \infty \quad \text{supp } f \cap \Gamma = \emptyset \right\}$$

and consider the unique self–adjoint operator associated with $\mathcal{H}^1(\Omega)$ this operator will be said to be the Laplacian on Ω with Dirichlet boundary condition on Γ and Neumann boundary condition on $\partial\Omega - \Gamma$.

For $u \in \mathcal{D}(-\Delta_N^\Omega)$, we have the a–priori information $u \in \mathcal{H}^1(\Omega)$ and $\Delta u \in L_2(\Omega)$, but for irregular domains it may be very hard or impossible to obtain useful bounds for $\sup |u|$. In Lemmas 1.1 and 1.2 we shall show, instead, that control of the $\mathcal{H}^1$–norm gives certain precise bounds on

$$\int_S u(x, y)\, dx\, dy$$

for small rectangles S (we are in $\mathbf{R}^2$ now). These bounds will subsequently play the role of a weak substitute for a Dirichlet boundary condition in Lemma 1.3.

Lemma 1.1. *Consider a rectangle $R \subset \mathbf{R}^2$, $R = (0,l) \times (0,h)$, made up of two adjacent subrectangles, $R_1 = (0,l)\times(0,w)$ and $R_2 = (0,l)\times(w,h)$, where $0 < w < h$. Then, for $u \in \mathcal{H}^1(R)$ we have*

$$\left| \frac{1}{w} \int_{R_1} u\, dx\, dy - \frac{1}{h-w} \int_{R_2} u\, dx\, dy \right| \leq (hl)^{1/2} \, \|\nabla u\|$$

Proof. Since $C^\infty(R)$ is dense in $\mathcal{H}^1(R)$ and R is convex, it is clear that we can assume u to be $C^\infty(\bar{R})$, without loss of generality.
Define

$$f(y) = \int_0^l u(x,y)\, dx$$

and

$$m_1 = \int_0^w f(y)\, dy = \int_{R_1} u(x,y)\, dx\, dy$$
$$m_2 = \int_w^h f(y)\, dy = \int_{R_2} u(x,y)\, dx\, dy$$

Clearly, there are points $y_1 \in [0,w]$ and $y_2 \in [w,h]$ such that $f(y_1) = m_1/w$, $f(y_2) = m_2/(h-w)$, and it follows that

$$\left| \frac{m_1}{w} - \frac{m_2}{h-w} \right| = \left| \int_{y_1}^{y_2} f'(y)\, dy \right| \leq \int_{y_1}^{y_2} \int_0^l |u_y(x,y)|\, dx\, dy \leq (hl)^{1/2} \, \|\nabla u\|$$

as claimed. QED

The important point in the following simple lemma is to have the powers of w and l in the asymmetric version $w \cdot l^{1/2}$.

Lemma 1.2. *Let $Q = (0,r) \times (0,h)$ and let $S = (0,l) \times (0,w)$ (with $0 < l \leq r$ and $0 < w \leq h/2$) be a subrectangle of Q.*
Then, for $u \in \mathcal{H}^1(Q)$ we have

$$\left| \int_S u\, dx\, dy \right| \leq C\, w\, l^{1/2} \, \|u\|_{\mathcal{H}^1(Q)}$$

with a constant C depending on h only.

Proof. Let

$$\tilde{Q} = (0,l) \times [w,h) \qquad Q^* = S \cup \tilde{Q}$$

Applying Lemma 1.1 to Q^*, we obtain

$$\left| \frac{1}{w} \int_S u \right| \le (lh)^{1/2} \, \|\nabla u\|_{L_2(Q^*)} + \frac{1}{(h-w)} \left| \int_{\tilde{Q}} u \right|$$
$$\le C_h \, l^{1/2} \, \|u\|_{\mathcal{H}^1(Q^*)}$$

with $C_h = h^{1/2}(1 + 2/h)$, and the result follows. QED

In the subsequent Lemma we consider a passage $P_w = (0,L) \times (0,w)$ of fixed length L and width $w \ll L$, with two adjoining rectangles S_1 and S_2 of length $l = l_w = w^{1/2}$, $S_1 = (-l,0] \times (0,w)$, and $S_2 = [L, L+l) \times (0,w)$.
Let $-\Delta_{DN}^{P_w}$ denote the Laplacian on P_w with Dirichlet boundary conditions at the ends of P_w and Neumann on the long sides of P_w. Also, let

$$0 < \mu_1 \le \mu_2 \le \ldots \le \mu_k \le \ldots$$

denote the eigenvalues of $-\Delta_{DN}^{P_w}$, repeated according to their multiplicities, and let $\{\psi_j\}_{j\in\mathbf{N}}$ denote a complete orthonormal set of eigenfunctions satisfying

$$-\Delta_{DN}^{P_w}\psi_j = \mu_j \, \psi_j \qquad j = 1, 2, \ldots$$

Clearly, any function v in the form domain of $-\Delta_{DN}^{P_w}$ which is orthogonal to $\psi_1, \ldots, \psi_k$ satisfies the inequality

$$\|\nabla v\|^2 \ge \mu_{k+1} \, \|v\|^2$$

We wish to extend this property to a family of functions $v_w \in \mathcal{H}^1(S_1 \cup P_w \cup S_2)$ which do not really obey a Dirichlet boundary condition, but instead, satisfy the condition

$$\int_{S_q} v_w = O(w l^{1/2}) \qquad \text{as} \quad w \to 0$$

Lemma 1.3. *Let $k \in \mathbf{N}_0$ be fixed. Suppose we are given a family $\{v_w\}_{0<w\le 1}$ of functions $v_w \in \mathcal{H}^1(S_1 \cup P_w \cup S_2)$ which satisfy the following conditions:*

(i) $\|v_w\|^2_{\mathcal{H}^1(S_1\cup P_w \cup S_2)} \le C_0, \quad 0 < w \le 1,$ *for some constant C_0.*

(ii) $\langle v_w, \psi_j\rangle \to 0, \quad w \to 0, \quad j = 1, \ldots, k$

(iii) $\left|\int_{S_q} v_w\right| \le A\, w\, l^{1/2}, \quad q = 1, 2, \quad 0 < w \le 1,$ *for some constant A.*

Then, for any $\epsilon > 0$ there exists w_ϵ such that

$$\|\nabla v_w \lceil P_w\|^2 \ge \mu_{k+1} \|v_w \lceil P_w\|^2 - \epsilon \qquad 0 < w \le w_\epsilon$$

Remark. We will apply this lemma only in cases where $\langle v_w, \psi_j\rangle = 0, \quad j = 1, \ldots, k$.

Proof. Again, we may assume $v_w \in C^\infty(S_1 \cup P_w \cup S_2)$.
On $S_1 \cup P_w \cup S_2$ we define the functions $\tilde{v} = \tilde{v}_w$ by

$$\tilde{v}(x, y) = \frac{1}{w} \int_0^w v(x, z)\, dz$$

(Note that $\tilde{v}$ is constant in the y–direction.) We then have

$$\begin{aligned} \|\nabla \tilde{v} \lceil P_w\|^2 &\le \|\nabla v \lceil P_w\|^2 \\ \|\tilde{v} \lceil P_w\|^2 &\ge \|v \lceil P_w\|^2 - w^2 \|v \lceil P_w\|_1^2 \end{aligned} \tag{1.1}$$

In order to prove (1.1), expand $v \lceil P_w$ in terms of eigenfunctions of $-\Delta_N^{P_w}$

$$v \lceil P_w = \sum_{p,q \in \mathbf{N}_0} \hat{v}(p, q)\, e_{p,q}$$

where $\hat{v}(p, q) = \langle v, e_{p,q}\rangle$ and the $e_{p,q}$ are given by

$$w^{-1/2} L^{-1/2} \cos\Big(\frac{\pi p x}{L}\Big) \cos\Big(\frac{\pi q y}{w}\Big) \qquad p, q \in \mathbf{N}_0$$

with normalizing factor between $\frac{1}{2}$ and 2. Clearly,

$$(v - \tilde{v}) \lceil P_w = \sum_{\substack{p,q \in \mathbf{N}_0 \\ q \neq 0}} \hat{v}(p,q)\, e_{p,q}$$

and, since $v \lceil P_w \in \mathcal{H}^1(P_w) = \mathcal{Q}(-\Delta_N^{P_w})$,

$$\|\nabla v \lceil P_w\|^2 = \pi^2 \sum_{p,q \neq (0,0)} \left(\frac{p^2}{L^2} + \frac{q^2}{w^2} \right) |\hat{v}(p,q)|^2$$

so that

$$\|(v - \tilde{v}) \lceil P_w\|^2 \leq \frac{w^2}{\pi^2} \|\nabla v\|^2 \leq \frac{w^2}{\pi^2} C_0 \tag{1.2}$$

Next, assumption (*iii*) implies that there exist $-l < x_1 \leq 0$ and $L \leq x_2 < L + l$ such that

$$|\tilde{v}(x_i, y)| \leq A\, l^{-1/2} \qquad 0 \leq y \leq w$$

Let $\xi_1 = 0$ and $\xi_2 = L$, so that ξ_i is the x–coordinate of the left or right end of P_w. We then have

$$|\tilde{v}(\xi_i, y)| \leq \tilde{A}\, w^{-1/4} \qquad 0 \leq y \leq w \tag{1.3}$$

by the the following easy argument: Let $h_i = |\tilde{v}(x_i) - \tilde{v}(\xi_i)|$ for $i = 1, 2$, so that $|\tilde{v}(\xi_i, y)| \leq A\, l^{-1/2} + h_i$ for $0 \leq y \leq w$. Using the trivial inequality

$$|f(0) - f(t)|^2 \leq t \int_0^t f'(s)^2\, ds \qquad f \in \mathcal{H}^1(0,t)$$

we obtain

$$\int_{x_i}^{\xi_i} |\partial_x \tilde{v}(x,y)|^2\, dx \geq \frac{h_i^2}{l} \qquad 0 \leq y \leq w$$

implying

$$\|\nabla v\|^2 \geq \|\nabla \tilde{v}\|^2 \geq \int_0^w \int_{x_i}^{\xi_i} |\partial_x \tilde{v}(x,y)|^2\, dx\, dy \geq \int_0^w \frac{h_i^2}{l}\, dy \geq w\, h_i^2\, l^{-1}$$

so that $h_i^2 \leq C_0 l / w$, and (1.3) follows.

Now let $\phi = \phi(x,y)$ be the (affine) linear function on P_w which makes

$$(\tilde{v} - \phi)(\xi_i, y) = 0 \qquad i = 1,2 \qquad 0 \le y \le w$$

By (1.3) (and since the length of P_w is held fixed), we have

$$\|\phi\|_1^2 = O\left(w^{1/2}\right) \qquad w \to 0 \tag{1.4}$$

As $\tilde{v} - \phi$ belongs to the form domain $\mathcal{Q}(-\Delta_{DN}^{P_w})$ and satisfies

$$|\langle \tilde{v} - \phi, \psi_j \rangle| \le 2\epsilon \qquad j = 1, \dots, k \qquad 0 < w < w'_\epsilon$$

(by (ii) and (1.2)), we obtain

$$\|\nabla(\tilde{v} - \phi) \lceil P_w\|^2 \ge \mu_{k+1}\left(\|(\tilde{v} - \phi) \lceil P_w\|^2 - 4k\epsilon^2\right)$$

for $0 < w < w'_\epsilon$. Using (1.1) and (1.4), it finally follows that

$$\begin{aligned}\|\nabla v \lceil P_w\|^2 \ge \|\nabla \tilde{v} \lceil P_w\|^2 &\ge \|\nabla(\tilde{v} - \phi) \lceil P_w\|^2 - \epsilon \\ &\ge \mu_{k+1}\left(\|(\tilde{v} - \phi) \lceil P_w\|^2 - 4\epsilon\right) - \epsilon \\ &\ge \mu_{k+1}\|v \lceil P_w\|^2 + O(\epsilon)\end{aligned}$$

for $0 < w < w''_\epsilon$, and we are done. QED

We now consider two domains Ω_1 and Ω_2 in $\mathbf{R}^2$, $\Omega_1 \cap \Omega_2 = \emptyset$, with piecewise smooth boundaries and $\left(-\Delta_N^{\Omega_q} + 1\right)^{-1}$ compact, joined with a passage $P_w = (0, L) \times (-w, w)$, as shown in Figure 3. Note that we prefer P_w to be symmetric with respect to the x–axis, in this context. We require our domains Ω_1 and Ω_2 to satisfy the following two conditions:

1. $P_s \cap (\Omega_1 \cup \Omega_2) = \emptyset$
2. $(-s, 0) \times (-s, s) \subset \Omega_1$ and $(L, L+s) \times (-s, s) \subset \Omega_2$,

for some $s > 0$. We require conditions 1 and 2 because it simplifies notation, and because they are satisfied by the examples we try to understand. They could be relaxed to require only that the boundary of the domains be smooth (with non-zero x-derivative) around the points where the passage is attached, and that $\partial\Omega_q$ intersects the line segment $[0,L] \times \{0\}$ only once, for $q = 1, 2$.

Insert Figure 3 here.

We define $P'_w = [0,L] \times (-w, w)$,

$$\Omega_w = \Omega_1 \cup P'_w \cup \Omega_2\,, \qquad H_w = -\Delta_N^{\Omega_w}$$

and

$$\tilde{H}_w = -\Delta_N^{\Omega_1} \oplus -\Delta_{DN}^{P_w} \oplus -\Delta_N^{\Omega_2}$$

Let $\lambda_i = \lambda_i(w)$, for $i = 1, 2, \ldots$ denote the eigenvalues of H_w,

$$0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_i \leq \ldots$$

repeated according to their multiplicities, and let $\{\varphi_i\}$ denote an associated orthonormal basis of eigenfunctions, $\varphi_i = \varphi_i(w)$. Similarly, $\tilde{\lambda}_i = \tilde{\lambda}_i(w)$, $\tilde{\varphi}_i = \tilde{\varphi}_i(w)$ denote the eigenvalues and eigenfunctions of $\tilde{H}_w$. Associated with the $\tilde{\varphi}_i$ we also consider their extension $\tilde{\psi}_i \in \mathcal{H}^1(\Omega_w)$, defined as follows:

1. If $\tilde{\varphi}_i$ is an eigenfunction of $-\Delta_{DN}^{P_w}$, it will be extended as 0 inside Ω_q, $q = 1, 2$.

2. In the case where $\tilde{\varphi}_i$ is an eigenfunction of $-\Delta_N^{\Omega_1}$, the Sobolev Extension Theorem (see Gilbarg and Trudinger [10 ; Theorem 7.25]) allows us to extend $\tilde{\varphi}_i$ to a domain that contains $\Omega_1 \cup P_w$. (Note that in our example, since $\partial\Omega_1 \cap B_s(0)$ is a straight line, a simple reflexion argument would do). Let these extensions be g_i. Let $j(x)$, smooth, be such that $j(0) = 1$ and $j(L) = 0$. We then define
$$\tilde{\psi}_i(x,y) = \begin{cases} g_i(x,y) \cdot j(x) & \text{on } P_w \\ 0 & \text{on } \Omega_2 \end{cases}$$

   It is clear, by dominated convergence, that $\tilde{\psi}_i - \tilde{\varphi}_i$ goes to zero in $L_2(\Omega_w)$ and that $\left\| \tilde{\psi}_i \restriction P_w \right\|_{\mathcal{H}^1(P_w)} \to 0$, as $w \to 0$.

The eigenfunctions of $-\Delta_N^{\Omega_2}$ are dealt with in the same way.

It is our aim to show that the differences of eigenvalues $\lambda_i(w) - \tilde{\lambda}_i(w)$ (and eigenfunctions $\varphi_i(w) - \tilde{\varphi}_i$) go to zero as $w \to 0$. To avoid notational difficulties in the case of degenerate eigenvalues, we consider the spectral families $\{E_\lambda\}_{\lambda\in\mathbf{R}}$ and $\{\tilde{E}_\lambda\}_{\lambda\in\mathbf{R}}$ for H_w and $\tilde{H}_w$ and prove the following:

Proposition 1.4. *Suppose the above assumptions are satisfied, and let $\Lambda > 0$ and $\epsilon > 0$ be given. Then there exists w_ϵ such that*

$$\left\| E_\lambda(w) - \tilde{E}_\lambda(w) \right\| < \epsilon \qquad 0 < w < w_\epsilon$$

for all $\lambda < \Lambda$ satisfying $\operatorname{dist}(\lambda, \sigma(\tilde{H}_w)) \geq \epsilon$.

Proof. Let $0 = \tilde{\mu}_1 < \tilde{\mu}_2 < \cdots < \tilde{\mu}_k < \cdots$ denote the points of $\sigma(\tilde{H}_w)$ and let $k_0 \in \mathbf{N}$ be such that $\tilde{\mu}_{k_0} \leq \Lambda$, and $\tilde{\mu}_{k_0+1} > \Lambda$. Note first that for w small enough, $\tilde{\mu}_k < \Lambda + 1$ and $\tilde{\lambda}_i \leq \Lambda + 1$ are independent of w.[1] Without loss of generality, we may assume that $|\Lambda - \tilde{\mu}_{k_0}| > \epsilon$, $|\Lambda - \tilde{\mu}_{k_0+1}| > \epsilon$, and that ϵ is small enough so that the 2ϵ–balls around the points $\tilde{\mu}_1, \ldots, \tilde{\mu}_{k_0}$ do not intersect.

A. In this part of the proof we show that, for $\lambda \leq \Lambda$,

(i) $\dim \mathcal{R}(E_{\lambda+\epsilon-0}) \geq \dim \mathcal{R}(\tilde{E}_\lambda)$.

(ii) $\dim \mathcal{R}(E_{\lambda-\epsilon}) \leq \dim \mathcal{R}(\tilde{E}_\lambda)$.

for w small (here $\mathcal{R}(\tilde{E}_\lambda)$ denotes the range of $\tilde{E}_\lambda$).
Suppose $\tilde{\lambda}_1 \leq \ldots \leq \tilde{\lambda}_p \leq \lambda$, while $\tilde{\lambda}_{p+1} > \lambda$ so that $\mathcal{R}(\tilde{E}_\lambda)$ is spanned by $\tilde{\varphi}_1, \cdots, \tilde{\varphi}_p$.
Defining

$$\tilde{M} = \operatorname{span}\{\tilde{\psi}_1, \ldots, \tilde{\psi}_p\}$$

where the $\tilde{\psi}_i$ are the extensions of the functions $\tilde{\varphi}_i$ to all of Ω_w of the functions $\tilde{\varphi}_i$, we clearly have (for w sufficiently small)

$$\|\nabla\psi\|^2 \leq (\tilde{\lambda}_p + \epsilon/2)\, \|\psi\|^2 \qquad \psi \in \tilde{M}$$

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[1] Although this is not essential for the proof, it slightly simplifies the picture.

and

$$\dim \tilde{M} = \dim \mathcal{R}(\tilde{E}_\lambda) = p$$

Since $\tilde{M}$ is contained in the form–domain of H_w, min–max implies that H_w has at least p eigenvalues below $\lambda + \epsilon$, proving (i).
If (ii) were not true, we could find $u \in R(E_{\lambda-\epsilon})$, $\|u\| = 1$, u orthogonal to the range of $\tilde{E}_\lambda$, whence

$$\langle u, \tilde{\varphi}_i \rangle = 0 \qquad i = 1, \dots, p \tag{1.5}$$

Although u will not in general belong to the form domain of $\tilde{H}_w$, we nevertheless conclude from (1.5) that

$$\|\nabla u \lceil \Omega_1\|^2 \geq \tilde{\lambda}_{p+1} \|u \lceil \Omega_1\|^2$$

and similarly for Ω_2, while combining Lemma 1.2 and 1.3 yields an estimate

$$\|\nabla u \lceil P_w\|^2 \geq \tilde{\lambda}_{p+1} \|u \lceil P_w\|^2 - \epsilon/2$$

for w small; we therefore end up with

$$\|\nabla u\|^2 \geq \tilde{\lambda}_{p+1} \|u\|^2 - \epsilon/2 \tag{1.6}$$

for w small.
On the other hand, $u \in R(E_{\lambda-\epsilon})$ implies $\|\nabla u\|^2 \leq (\lambda - \epsilon) \|u\|^2$, in contradiction with (1.6) and $\tilde{\lambda}_{p+1} > \lambda$. This proves (ii).

B. Denote the points where the ϵ–balls around the $\tilde{\mu}_k$, $k \geq 1$, intersect the real line by $x_1 < \cdots < x_j < \dots$, so that $x_{2k} = \tilde{\mu}_k + \epsilon$, $x_{2k-1} = \tilde{\mu}_k - \epsilon$ and let $x_0 = -\infty$. Defining

$$\mathcal{P}_j = E_{x_j} - E_{x_{j-1}} \qquad \tilde{\mathcal{P}}_j = \tilde{E}_{x_j} - \tilde{E}_{x_{j-1}},$$

it is evident that $\mathcal{P}_1 = \tilde{\mathcal{P}}_1 = 0$ (as $x_1 = -\epsilon$ and $x_0 = -\infty$), and that $\tilde{\mathcal{P}}_j = 0$ for j odd. Applying (i) and (ii) succesively at the points x_j, it is easy to see that all eigenvalues of H_w in $[0, \Lambda]$ lie inside the intervals $(\tilde{\mu}_k - \epsilon, \tilde{\mu}_k + \epsilon)$, $k = 1, \dots, k_0$; furthermore, if $\mu \in \sigma(\tilde{H}_w)$, $\mu \leq \Lambda$, is an eigenvalue of multiplicity m, then there are precisely m eigenvalues of H_w inside the 2ϵ–interval centered at this μ, counting multiplicity.

C. Finally, we are in a position to prove that, for w small,

$$\left\| \mathcal{P}_j - \tilde{\mathcal{P}}_j \right\| < \epsilon \qquad 1 \leq j \leq 2k_0.$$

There is nothing to be proven for $j = 1$, and we assume now that the assertion holds for $1, \ldots, j-1 < 2k_0$. By part B of our proof, we have

$$\dim R(\mathcal{P}_j) = \dim R(\tilde{\mathcal{P}}_j) \tag{1.7}$$

If j is odd, then $\tilde{\mathcal{P}}_j = 0$ and also $\mathcal{P}_j = 0$, by (1.7), and we are done.
If j is even, let $m = \dim \mathcal{R}(\tilde{\mathcal{P}}_j)$ and suppose that $\tilde{\lambda}_p = \cdots = \tilde{\lambda}_{p+m-1}$ are in the interval (x_{j-1}, x_j), so that $\lambda_p, \ldots, \lambda_{p+m-1}$ lie in the interval (x_{j-1}, x_j) while $\lambda_{p+m} \geq \tilde{\lambda}_{p+m} - \epsilon$, by part (B). Then

$$\tilde{\lambda}_p = \|\nabla \tilde{\varphi}_p\|^2 = \left\|\nabla \tilde{\psi}_p\right\|^2 + o(1) \qquad w \to 0$$

where (noting that $\tilde{\psi}_p \in \mathcal{H}^1(\Omega_w) = \mathcal{D}(H_w^{1/2})$)

$$\left\|\nabla \tilde{\psi}_p\right\|^2 = \left\|H_w^{1/2} \tilde{\psi}_p\right\|^2 = \int_{-\infty}^{\infty} \lambda \, d \left\| E_\lambda \tilde{\psi}_p \right\|^2$$

By the induction hypothesis, we know that $\left\| E_{x_{j-1}} \tilde{\psi}_p \right\|^2 \to 0$, as $w \to 0$, and it follows that

$$\tilde{\lambda}_p \geq \lambda_p \left\| \mathcal{P}_j \tilde{\psi}_p \right\|^2 + \lambda_{p+m} \sum_{l>j} \left\| \mathcal{P}_l \tilde{\psi}_p \right\|^2 + o(1)$$

Letting

$$d = \min\{ |\tilde{\mu}_k - \tilde{\mu}_l| \mid 1 \leq k, l \leq k_0, \ k \neq l \}$$

so that $\lambda_{p+m} \geq \lambda_p + d + 2\epsilon$, we may conclude that

$$(d - 2\epsilon) \sum_{l>j} \left\| \mathcal{P}_l \tilde{\psi}_p \right\|^2 \leq \lambda_p + \epsilon - \lambda_p \sum_{l \geq j} \left\| \mathcal{P}_l \tilde{\psi}_p \right\|^2 + o(1) \leq \epsilon + o(1) \ , w \to 0$$

as $\sum_{l \geq j} \left\| \mathcal{P}_l \tilde{\psi}_p \right\|^2 = 1 + o(1)$. This implies $\|\tilde{\varphi}_p - \mathcal{P}_j \tilde{\varphi}_p\|^2 \leq 2\epsilon/d + o(1)$, as $w \to 0$, since $\epsilon \leq d/4$. Repeating the same argument for the eigenfunctions $\tilde{\varphi}_{p+1}, \ldots, \tilde{\varphi}_{p+m-1}$ and using (1.7), it is easy to see that $\left\| \mathcal{P}_j - \tilde{\mathcal{P}}_j \right\| < \epsilon$ for w small enough. QED

Using Proposition 1.4, it is now straightforward to estimate the difference of the resolvents of H_w and $\tilde{H}_w$.

Proposition 1.5. *Let H_w and $\tilde{H}_w$ as before, and assume that $\left(-\Delta_N^{\Omega_q}+1\right)^{-2} \in \mathcal{B}_1$, for $q=1,2$. Then*

$$\left\|(H_w+1)^{-1}-(\tilde{H}_w+1)^{-1}\right\| \to 0 \qquad w \to 0$$

and

$$\left\|(H_w+1)^{-2}-(\tilde{H}_w+1)^{-2}\right\|_{\mathcal{B}_1} \to 0 \qquad w \to 0$$

where $\|\cdot\|_{\mathcal{B}_1}$ denotes the trace norm.

Proof.
As in the proof of Proposition 1.4, let $0=\tilde{\mu}_1<\tilde{\mu}_2<\ldots$ denote the points of $\sigma(\tilde{H}_w)$ and let $x_{2k}=\tilde{\mu}_k+\epsilon$, $x_{2k-1}=\tilde{\mu}_k-\epsilon$, $\mathcal{P}_j=E_{x_j}-E_{x_{j-1}}$, $\tilde{\mathcal{P}}=\tilde{E}_{x_j}-\tilde{E}_{x_{j-1}}$. Also let $\lambda_i^*(w)$ denote the (repeated) eigenvalues of $-\Delta_N^{\Omega_1}\oplus-\Delta_N^{P_w}\oplus-\Delta_N^{\Omega_2}$. By assumption, $(-\Delta_N^{\Omega_q}+1)^{-2}\in\mathcal{B}_1$, and, by inspection, $(-\Delta_N^{P_s}+1)^{-2}\in\mathcal{B}_1$. Since the eigenvalues of $-\Delta_N^{P_w}$ are monotonically non-decreasing, as $w\to 0$, and since $\lambda_i(w)\geq\lambda_i^*(w)$, $\tilde{\lambda}_i(w)\geq\tilde{\lambda}_i^*(w)$, it follows that for $\epsilon>0$ given, there exists $\Lambda>0$ such that $1/\Lambda<\epsilon$ and

$$\sum_{\lambda_i>\Lambda}(\lambda_i(w)+1)^{-2}+\sum_{\tilde{\lambda}_i>\Lambda}(\tilde{\lambda}_i(w)+1)^{-2}<\epsilon \tag{1.8}$$

for $0<w\leq s$. Now let $w_\epsilon>0$ so small that the $\tilde{\lambda}_i(w)\in[0,\Lambda+1]$ are independent of w, for $0<w\leq w_\epsilon$ and suppose that $p\in\mathbf{N}$ is such that $\tilde{\lambda}_p(w)\leq\Lambda$, while $\tilde{\lambda}_{p+1}>\Lambda$, for $0<w\leq w_\epsilon$. Finally, let $K\in\mathbf{N}$ be such that $\tilde{\mu}_{K-1}\leq\Lambda$, while $\tilde{\mu}_K>\Lambda$. Without restriction, we may also assume that $\operatorname{dist}(\Lambda,\tilde{\mu}_k)>\epsilon$, for $k=1,2,\ldots$.
For the first statement,

$$\begin{aligned}\left\|(H_w+1)^{-1}-(\tilde{H}_w+1)^{-1}\right\|& \\ &\leq\sum_{j=1}^{2K}\left\|\mathcal{P}_j(H_w+1)^{-1}-\tilde{\mathcal{P}}_j(\tilde{H}_w+1)^{-1}\right\| \\ &\quad+\left\|(1-E_{x_{2K}})(H_w+1)^{-1}\right\|+\left\|(1-\tilde{E}_{x_{2K}})(\tilde{H}_w+1)^{-1}\right\|\end{aligned}$$

Here the last two terms are bounded by $2\Lambda^{-1} < 2\epsilon$. In the sum, the contributions coming from j odd are zero. For j even, $j = 2k$, say, note that $(x_{j-1}, x_j) = (\tilde{\mu}_k - \epsilon, \tilde{\mu}_k + \epsilon)$, so

$$\tilde{\mathcal{P}}_j(\tilde{H}_w + 1)^{-1} = (\tilde{\mu}_k + 1)^{-1}\tilde{\mathcal{P}}_j$$

and

$$\left\|\mathcal{P}_j(H_w + 1)^{-1} - (\tilde{\mu}_k + 1)^{-1}\mathcal{P}_j\right\| \leq \epsilon/(1-\epsilon)$$

so that ($\epsilon \leq 1/2$, without restriction)

$$\left\|\mathcal{P}_j(H_w + 1)^{-1} - \tilde{\mathcal{P}}_j(\tilde{H}_w + 1)^{-1}\right\| \leq \left\|\mathcal{P}_j - \tilde{\mathcal{P}}_j\right\| + 2\epsilon$$

Using Proposition 1.4, it is now easy to obtain the first statement. For the second statement, we proceed in a similar way:

$$\begin{aligned}
&\left\|(H_w + 1)^{-2} - (\tilde{H}_w + 1)^{-2}\right\|_{\mathcal{B}_1} \\
&\qquad \leq \sum_{j=1}^{2K} \left\|\mathcal{P}_j(H_w + 1)^{-2} - \tilde{\mathcal{P}}_j(\tilde{H}_w + 1)^{-2}\right\|_{\mathcal{B}_1} \\
&\qquad\quad + \left\|(1 - E_{x_{2K}})(H_w + 1)^{-2}\right\|_{\mathcal{B}_1} + \left\|(1 - \tilde{E}_{x_{2K}})(\tilde{H}_w + 1)^{-2}\right\|_{\mathcal{B}_1}
\end{aligned}$$

Here the last two terms are less than ϵ, by (1.8). In the sum, we again have to consider j even only, where we now estimate

$$\begin{aligned}
&\left\|\mathcal{P}_j(H_w + 1)^{-2} - \tilde{\mathcal{P}}_j(\tilde{H}_w + 1)^{-2}\right\|_{\mathcal{B}_1} \\
&\qquad \leq \left\|\mathcal{P}_j - \tilde{\mathcal{P}}_j\right\|_{\mathcal{B}_1} + 2\epsilon \left\|\mathcal{P}_j\right\|_{\mathcal{B}_1} \leq 2 \dim \mathcal{R}(\mathcal{P}_j) \left\|\mathcal{P}_j - \tilde{\mathcal{P}}_j\right\| + 2\epsilon \dim \mathcal{R}(\mathcal{P}_j) \\
&\qquad \leq 2p \left\|\mathcal{P}_j - \tilde{\mathcal{P}}_j\right\| + 2\epsilon p \leq 4\epsilon p
\end{aligned}$$

for w small, by Propostion 1.4, and the result follows.

QED

On the intervals of length w, where the passage P_w meets the rooms Ω_q, the decoupled operator $\tilde{H}_w$ has Neumann boundary conditions from the side of the rooms and Dirichlet boundary condition from the side of the passage. As we will see now, we might as well decouple with a *pure* Dirichlet boundary condition on these intervals. In view of later applications, we consider the Neumann Laplacian on a domain Ω and investigate the influence of a Dirichlet boundary condition on the interval

$$I_\delta = [0, \delta] \times \{0\} \subset \mathbf{R}^2 \qquad \delta \geq 0 \quad ;$$

note that we do *not* require $I_\delta \subset \partial\Omega$. We have:

Proposition 1.6. *Suppose Ω is an open subset of $\mathbf{R}^2$ with $(-\Delta_N^\Omega + 1)^{-1}$ compact. Let $-\Delta_{D_\delta N}^\Omega$ denote the Laplacian on $\Omega - I_\delta$, with Dirichlet boundary condition on I_δ, and Neumann boundary conditions on the remaining portions of $\partial\Omega$. Then, given $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that*

$$\left\| \left(-\Delta_{D_\delta N}^\Omega + 1\right)^{-1} - \left(-\Delta_N^\Omega + 1\right)^{-1} \right\| < \epsilon \qquad 0 < \delta \leq \delta_\epsilon$$

Proof. Suppose we associate the objects λ_i, φ_i, μ_k, E_λ with the operator $-\Delta_N^\Omega$, and $\tilde{\lambda}_i$ etc. with $-\Delta_{D_\delta N}^\Omega$, as in the proof of Proposition 1.4. While the basic strategy of proof is the same as the one leading to Propositions 1.4 and 1.5, we now use the μ_k's instead of the $\tilde{\mu}_k$'s as reference points, the μ_k being independent of δ. Furthermore, there are substantial simplifications in the details; in fact, we shall need neither Lemma 1.3 nor the extension process $\tilde{\varphi}_i \mapsto \tilde{\psi}_i$. In particular, the estimate

$$(i) \qquad \dim \mathcal{R}(E_\lambda) \geq \dim \mathcal{R}(\tilde{E}_\lambda) \qquad \lambda \in \mathbf{R}$$

is now an immediate consequence of the fact that $-\Delta_N^\Omega \leq -\Delta_{D_\delta N}^\Omega$, in the sense of quadratic forms. Let $\epsilon > 0$. To obtain the estimate

$$(ii) \qquad \dim \mathcal{R}(E_\lambda) \leq \dim \mathcal{R}(\tilde{E}_{\lambda+\epsilon}) \qquad \lambda \leq \Lambda$$

for δ small, suppose that $\varphi_1, \ldots, \varphi_p$ span $\mathcal{R}(E_\lambda)$. Applying Lemma 1.7, below, to $\varphi_1, \ldots, \varphi_p$, we obtain functions $\psi_1, \ldots, \psi_p \in \mathcal{Q}(-\Delta_{D_\delta N}^\Omega)$ which satisfy

$$\|\varphi_i - \psi_i\|_1 \to 0 \qquad \delta \to 0$$

Letting $M = \text{span}\{\psi_1, \ldots, \psi_p\}$, we again see that $\dim M = p$ and that

$$\|\nabla u\|^2 \leq (\lambda + \epsilon) \|u\|^2 \qquad u \in M$$

for δ suficiently small, and (ii) follows. This corresponds to part (A) of the proof of Proposition 1.4. Applying the above estimates successively to the points $\mu_k \pm \epsilon$, we see that the eigenvalues of $-\Delta^{\Omega}_{D_\delta N}$ lie in the ϵ-neighborhood of the eigenvalues of $-\Delta^{\Omega}_{N}$. The argument given in part (C) of the proof of Proposition 1.4 is slightly simplified as $\tilde{\varphi}_p \in \mathcal{Q}(-\Delta^{\Omega}_{N})$, so that the proof can start from

$$\tilde{\lambda}_p = \|\nabla \tilde{\varphi}_p\|^2 = \int \lambda d \|E_\lambda \tilde{\varphi}_p\|^2$$

The rest of the arguments used in proving Propositions 1.4 and 1.5 remains basically unchanged.

QED

Lemma 1.7. *Let $\Omega \subset \mathbf{R}^2$ open and $u \in \mathcal{H}^1(\Omega)$. Then, there exists a sequence $\{u_n\} \subset \mathcal{H}^1(\Omega)$ such that u_n vanishes on the ball of radius $1/n$, centered a the origin, and $\|u - u_n\|_1 \to 0$ as $n \to \infty$.*

Proof.
For $\epsilon > 0$ given, there exists $M > 0$ such that the function $u_M \in \mathcal{H}^1(\Omega)$ given by

$$u_M(x,y) = \begin{cases} u(x,y) & \text{if } |u(x,y)| \leq M \\ u(x,y)/|u(x,y)| & \text{otherwise} \end{cases}$$

satisfies $u_M \in \mathcal{H}_1(\Omega)$ and $\|u - u_M\|_1 < \epsilon$. (This follows easily from the chain rule (Gilbarg and Trudinger [10; Theorem 7.8]) via dominated convergence.)
Now, let $\varphi \in C^\infty(\mathbf{R}^2)$ enjoy the properties $0 \leq \varphi \leq 1$, $\varphi(x,y) = 1$ if $x^2 + y^2 \geq 4$, and $\varphi(x,y) = 0$ for $x^2 + y^2 \leq 1/n^2$. Also, define

$$\varphi_k(x,y) = \varphi(kx, ky) \qquad k \in \mathbf{N}$$

From

$$\|u_M - \varphi_k u_M\| \to 0 \qquad k \to \infty$$

and $\|\nabla(\varphi_k u_M)\| \leq$ const., it follows that there exists a sequence $\{k_j\} \subset \mathbf{N}$ such that $\varphi_{k_j} u_M \to u_M$ weakly in $\mathcal{H}^1(\Omega)$. Therefore, the Banach–Saks theorem implies that

$$\frac{1}{N} \sum_{j=1}^{N-1} \varphi_{k_j} u_M \to u_M \ , \quad \text{strongly in } \mathcal{H}^1(\Omega)$$

as $N \to \infty$, and the result follows. QED

This concludes the preparations needed for Section 2. We note at this point, that our proofs can be easily modified to obtain results similar to Propositions 1.4 – 1.6 in higher dimensions.
For the construction of various combs in Section 3, we shall need variants of Lemma 1.3 and Propositions 1.4 and 1.5 adapted to the situation where a thin "tooth" or handle is attached to a given domain. In view of Theorem 3.7, we will allow for the handles to be slightly more general than mere rectangles.

We begin with a variant of Lemma 1.3, dealing with a family of handles D_w, $0 < w \leq w_0$, which are of the following type: for $0 < w \leq w_0$, each D_w is a bounded, open, connected subset of the right half plane in $\mathbf{R}^2$, satisfying

$$D_w \cap \{(x,y) \in \mathbf{R}^2 \mid 0 < x < \sqrt{w}\} = (0, \sqrt{w}) \times (0, w) \qquad 0 < w \leq w_0 \tag{1.9}$$

(This means that D_w begins with an actual rectangular handle on the left–hand side). Letting $-\Delta_{DN}^{D_w}$ denote the Laplacian on D_w with Dirichlet boundary conditions on $\{0\} \times [0,w]$ and Neumann boundary conditions everywhere else, we require that $(-\Delta_{DN}^{D_w} + 1)^{-1}$ is compact.
Also, let $0 < \mu_1 \leq \ldots \leq \mu_j \leq \ldots$ denote the (repeated) eigenvalues of $-\Delta_{DN}^{D_w}$, with associated normalized eigenfunctions ψ_j, $j = 1, 2, \ldots$. Finally, we need the adjoining rectangles $S_w = (-\sqrt{w}, 0] \times (0, w)$.

With the above notation and assumptions we have the following lemma.

Lemma 1.8. *Let $k \in \mathbf{N}_0$, $w_0 > 0$, and suppose we are given a family $\{v_w\}_{0<w\le w_0}$, with $v_w \in \mathcal{H}^1(D_w \cup S_w)$ satisfying the following conditions:*

(i) $\|v_w\|_{\mathcal{H}^1(D_w\cup S_w)} \le C$ *for* $0 < w \le w_0$ *and some constant* C.

(ii) $\langle \psi_j, v_w\rangle \to 0$, *as* $w \to 0$, *for* $j = 1, \dots, k$.

(iii) $|\int_{S_w} v_w| \le A w^{5/4}$, *for some constant* A *and* $0 < w \le w_0$.

Then, for any $\epsilon > 0$ *there exists* w_ϵ *such that*

$$\|\nabla v_w \restriction D_w\|^2 \ge \mu_{k+1} \|v_w \restriction D_w\|^2 - \epsilon \qquad 0 < w \le w_\epsilon$$

Proof.
The simple averaging used in the proof of Lemma 1.3 has to be refined. We write $v = v_w$, $l = \sqrt{w}$ and use the notation

$$h_w = (-l, l) \times (0, w) \subset \mathbf{R}^2$$

$$e_{p,q}(x,y) = c_{p,q} \cos\frac{\pi p x}{2l} \cos\frac{\pi q y}{w} \qquad (x,y) \in h_w$$

for $p, q \in \mathbf{N}_0$ and suitable normalizing constants $c_{p,q}$; also, put $\hat{v}_{p,q} = \langle v, e_{p,q}\rangle$. Writing

$$U_0 = \sum_{\substack{p\in\mathbf{N}_0\\ q=0}} \hat{v}_{p,q} e_{p,q} \quad , \qquad U_1 = v \restriction h_w - U_0 = \sum_{\substack{p\in\mathbf{N}_0\\ q\in\mathbf{N}}} \hat{v}_{p,q} e_{p,q}$$

we now define, for $(x,y) \in D_w$

$$\tilde{v}(x,y) = \begin{cases} v(x,y) & x \ge \sqrt{w} \\ U_0(x,y) + \phi_w(x) U_1(x,y) & -\sqrt{w} \le x \le \sqrt{w} \end{cases}$$

where $\phi_w(x) \in C^\infty(\mathbf{R})$ is such that $\phi_w(x) = 1$ for $x \ge \sqrt{w}$, $\phi_w(x) = 0$ for $x \le 0$, $0 \le \phi_w \le 1$ and $\max|\phi_w'| \le 2\, w^{-1/2}$.

As in the proof of Lemma 1.3, the assumption $\|\nabla v\|^2 \le C$ implies $\|U_1\|^2 = O(w^2)$ for w small, so that $\|v - \tilde{v}\| \to 0$. To estimate $\|\nabla\tilde{v}\|^2$, we first observe that $\partial_y U_0 = 0$, so that

$$\int_{h_w} |\partial_y \tilde{v}|^2 \le \int_{h_w} |\phi_w|^2 |\partial_y U_1|^2 \le \int_{h_w} |\partial_y v|^2$$

On the other hand, looking at the x–derivative $\partial_x \tilde{v} = \partial_x U_0 + \phi_w'(x) U_1 + \phi_w(x) \partial_x U_1$, we first note that

$$\int_{h_w} |\phi_w'(x)|^2 \, |U_1|^2 \le \max |\phi_w'|^2 \, \|U_1\|^2 \le C\, w^{-1} w^2$$

Furthermore, as $q = 0$ in U_0 and $q \ge 1$ in U_1, Fubini's theorem implies $\langle \partial_x U_0, \partial_x U_1 \rangle = 0$ and it follows that

$$\begin{aligned}\int_{h_w} |\partial_x \tilde{v}|^2 &= \int_{h_w} |\partial_x U_0|^2 + \int_{h_w} |\phi_w(x)|^2 \, |\partial_x U_1|^2 + O\left(w^{1/2}\right) \\ &\le \int_{h_w} |\partial_x v|^2 + O\left(w^{1/2}\right)\end{aligned}$$

We therefore conclude that $\|\nabla \tilde{v}\|^2 \le \|\nabla v\|^2 + O\left(w^{1/2}\right)$. The rest of the proof is similar to the proof of Lemma 1.3; here, however, we will have to subtract from $\tilde{v}$ a piecewise linear function which is 0 for $x \ge \sqrt{w}$. QED

We now join handles of the type described above to a fixed domain $\Omega \subset \mathbf{R}^2$, where we assume that

$$(-s_0, 0) \times (-s_0, s_0) \subset \Omega$$

for some $s_0 > 0$, and

$$\Omega \cap D_w = \emptyset \qquad 0 < w \le w_0$$

We will also require

$$D_w \cap \{(x,y) \mid 0 < x < w^{1/4}\} = (0, w^{1/4}) \times (0, w)$$

which implies (1.9), for $w \le 1$.

Now, letting $\Omega_w = \Omega \cup D_w'$, where $D_w' = D_w \cup (\{0\} \times (0, w))$, we define $H_w = -\Delta_N^{\Omega_w}$ and $\tilde{H}_w = -\Delta_N^{\Omega} \oplus -\Delta_{DN}^{D_w}$. Also, let $\lambda_i(w)$, $i = 1, 2, \ldots$, denote the eigenvalues of H_w, repeated according to multiplicity. We have

Proposition 1.9. *In addition to the above assumptions, suppose that* $(-\Delta_N^\Omega + 1)^{-2} \in \mathcal{B}_1$ *and that, more strongly,*

$$\sum_{i \geq N} (\lambda_i(w) + 1)^{-2} \to 0 \qquad N \to \infty$$

uniformly for $0 < w \leq w_0$. *Finally, assume that, for any* $\Lambda > 0$ *given, the eigenvalues of* $-\Delta_{DN}^{D_w}$ *in* $[0, \Lambda]$ *are independent of* w, *for* w *small enough.*
Then, as $w \to 0$,

$$\left\| (H_w + 1)^{-1} - (\tilde{H}_w + 1)^{-1} \right\| \to 0$$

and

$$\left\| (H_w + 1)^{-2} - (\tilde{H}_w + 1)^{-2} \right\|_{\mathcal{B}_1} \to 0$$

Proof. Using Lemma 1.8 in place of Lemma 1.3, we closely follow the strategy of proof which led to Propositions 1.4 and 1.5. Note that D_w now has a rectangular handle $(0, w^{1/4}) \times (0, w)$, so that we can use the old extension and cut–off process to extend the eigenfunctions $\tilde{\varphi}_i$ of $\tilde{H}_w$ to functions $\tilde{\psi}_i$ defined on all of Ω_w, $\tilde{\psi}_i \in \mathcal{H}^1(\Omega_w)$, and such that $\left\| \tilde{\psi}_i \lceil D_w \right\|_{\mathcal{H}^1(D_w)} \to 0$, as $w \to 0$. QED

Remarks. (a)The assumption that the eigenvalues of $-\Delta_{DN}^{D_w}$ in any interval $[0, \Lambda]$ be independent of w, for w small, is very restrictive. However, it is easy to see that the proof of Proposition 1.9 can be modified to cover the situation where, for any i, the eigenvalue $\tilde{\lambda}_i(w)$ of $-\Delta_{DN}^{D_w}$ converges to some limit $\hat{\lambda}_i$, as $w \to 0$, with $\hat{\lambda}_i \to \infty$ as $i \to \infty$.

(b) Some related results on shrinking handles attached to a fixed domain can be found in Courant and Hilbert [5 ; p. 420] and in Arrieta, Hale and Han [3].

2. Rooms and Passages

In Section 1, we considered the Neumann Laplacian on domains consisting of two rooms, joined by a narrow passage. We now analyze the case where an infinite number of rooms are joined by narrow passages and we determine the essential spectrum of the associated Neumann Laplacian. More specific results will be obtained by choosing the rooms to be either rectangles (Corollary 2.2) or rectangles with a partition (Corollaries 2.4 and 2.5). Rooms with partitions are particularly useful to attack the inverse problem

(IP) Given a closed set $S \subset [0,\infty)$, does there exist a bounded domain $\Omega \subset \mathbf{R}^2$ such that $\sigma_{\rm ess}(-\Delta_N^\Omega) = S$?

We now define the general setting for rooms and passages (cf. Figure 4):
Suppose we are given two bounded, strictly increasing sequences $\{x_k\}$, $\{x'_k\} \subset [0,\infty)$ which interlace, in the sense that $x_k < x'_k < x_{k+1}$, for $k = 1, 2, \ldots$. We also assume $x_1 = 0$, for simplicity. For $k = 1, 2, \ldots$, let $R_k \subset \mathbf{R}^2$ be open sets satisfying the following three conditions:

$$(-\Delta_N^{R_k} + 1)^{-2} \quad \text{is trace class.} \tag{2.1}$$

$$R_k \subset \{(x,y) \mid x_k < x < x'_k\} \tag{2.2}$$

$$\partial R_k \cap B_{\epsilon_k}((x_k,0)) = \{x_k\} \times (-\epsilon_k,\epsilon_k) \qquad \partial R_k \cap B_{\epsilon_k}((x'_k,0)) = \{x'_k\} \times (-\epsilon_k,\epsilon_k) \tag{2.3}$$

for some $\epsilon_k > 0$, where $B_{\epsilon_k}((x_k,0))$ denotes the ball of radius ϵ_k centerd at the point $(x_k,0)$. Conditions (2.2) and (2.3) in particular imply that the right half of $B_{\epsilon_k}((x_k,0))$ and the left half of $B_{\epsilon_k}((x'_k,0))$ are contained in R_k.

Insert Figure 4.

We then define the k-th passage, joining R_k with R_{k+1}, by

$$P_k = P_k(w_k) = (x'_k, x_{k+1}) \times (-w_k, w_k) \qquad 0 < w_k < \min(\epsilon_k, \epsilon_{k+1}) \tag{2.4}$$

While the rooms R_k may be considered as being fixed, the widths w_k will be determined later on. For a sequence $\{w_k\}$ satisfying the requirements in (2.4), we now define the rooms and passages domain Ω by

$$\Omega = \Omega\left(\{w_k\}_{k\in\mathbf{N}}\right) = \cup_{k=1}^{\infty}\left(R_k \cup P'_k\right) \tag{2.5}$$

where $P'_k = [x'_k, x_{k+1}] \times (-w_k, w_k)$. Also, define the domains obtained by joining the first n rooms

$$\Omega_n = \Omega_n\left(\{w_k\}_{k=1,\dots,n-1}\right) = \cup_{k=1}^{n-1}\left(R_k \cup P'_k\right) \cup R_n \tag{2.6}$$

and the approximating operators

$$H_n = H_n(\{w_k\}_{k\in\mathbf{N}}) = -\Delta_N^{\Omega_n} \oplus \left(\bigoplus_{k=n}^{\infty} (-\Delta_{DN}^{P_k} \oplus -\Delta_N^{R_{k+1}})\right) \tag{2.7}$$

$$\tilde{H}_n = \tilde{H}_n(\{w_k\}_{k\in\mathbf{N}}) = -\Delta_{DN}^{\Omega_n} \oplus \left(\bigoplus_{k=n}^{\infty} (-\Delta_{DN}^{P_k} \oplus -\Delta_{DN}^{R_{k+1}})\right) \tag{2.8}$$

Here the boundary conditions for $-\Delta_{DN}^{P_k}$ are as in Section 1, while the Laplacian $-\Delta_{DN}^{\Omega_n}$ obeys Dirichlet boundary conditions on the line segment where the passage P_n meets Ω_n, and Neumann conditions on the remaining parts of the boundary $\partial\Omega_n$. Similarly, $-\Delta_{DN}^{R_k}$ has Dirichlet boundary conditions on the line segments where the passages P_{k-1} and P_k are attached, and Neumann boundary conditions on the rest of ∂R_k. Hence $\tilde{H}_n$ is the Neumann Laplacian on Ω, with all but the first n rooms and all but the first $n-1$ passages decoupled by pure Dirichlet boundary conditions. (Note that the meaning of the "$\tilde{}$" differs from Section 1).
The fundamental result of this section reads as follows.

Theorem 2.1. *Suppose we are given $R_k \subset \mathbf{R}^2$, $k = 1, 2, \dots$, satisfying conditions (2.1)—(2.3). Then, there exists a sequence of positive numbers w_k, $w_k \to 0$ as $k \to \infty$, such that the Neumann Laplacian on $\Omega = \Omega\left(\{w_k\}\right)$ enjoys the following properties:*

(i) $\sigma_{\mathrm{ac}}(-\Delta_N^{\Omega}) = \emptyset$.

(ii) $\sigma_{\mathrm{ess}}(-\Delta_N^{\Omega}) = \sigma_{\mathrm{ess}}\left(\bigoplus_{k=1}^{\infty} - \Delta_N^{R_k}\right) = \bigcap_{n\geq 1}\left(\bigcup_{k\geq n}\sigma(-\Delta_N^{R_k})\right)^{\text{Closure}}$

Remarks. (a) Any isolated point of $\sigma_{\rm ess}(-\Delta_N^\Omega)$ is an eigenvalue of infinite multiplicity or an accumulation point of eigenvalues. In particular, if 0 is an isolated point of $\sigma_{\rm ess}(-\Delta_N^\Omega)$, then it is necessarily an accumulation point of eigenvalues.

(b) The result of Theorem 2.1 holds true for all sequences $\{w_k\}$ wich tend to 0 fast enough; cf. Theorem A.1 in the Appendix.

(c) It has been known for some time that rooms and passages examples may have a non-compact embedding of $\mathcal{H}^1(\Omega)$ into $L_2(\Omega)$ (cf. Courant and Hilbert [6 ; p. 521]), in which case the essential spectrum of the Neumann Laplacian can't be empty. More recently, Amick [2] and Evans and Harris [7, 8] analyzed various fundamental properties of rooms and passages type domains related to Poincaré's inequality and the measure of non-compactness of the embedding of $\mathcal{H}^1(\Omega)$ into $L_2(\Omega)$; they also determined the bottom of the essential spectrum in some cases.

Proof of Theorem 2.1.

(A) We first show that we can find a sequence $\{w_k\}$ of positive numbers such that

$$\left\|(H_n+1)^{-1}-(H_{n+1}+1)^{-1}\right\| \le \tfrac{1}{n^2} \qquad \left\|(H_n+1)^{-2}-(H_{n+1}+1)^{-2}\right\|_{\mathcal{B}_1} \le \tfrac{1}{n^2} \tag{2.9}$$

and

$$\left\|(H_n+1)^{-1}-(\tilde H_n+1)^{-1}\right\| \le \sum_{k\ge n} \tfrac{1}{k^2} \tag{2.10}$$

holds, for all $n = 1, 2, \ldots$.

To achieve this, we first apply Proposition 1.6 to all the rooms R_k to obtain a sequence $\{\overline{w}_k\}$, $\overline{w}_k > 0$, such that

$$\left\|(-\Delta_N^{R_k}+1)^{-1}-(-\Delta_{DN}^{R_k}+1)^{-1}\right\| \le \tfrac{1}{k^2} \tag{2.11}$$

provided $0 < w_k \le \overline{w}_k$ (recall that $-\Delta_{DN}^{R_k}$ obeys Dirichlet boundary conditions on the line segments $\{x_k\} \times (-w_{k-1}, w_{k-1})$ and on $\{x_k'\} \times (-w_k, w_k)$).

We now proceed by induction. For $n = 1$, (2.10) follows directly from (2.11), as H_1 and $\tilde H_1$ are fully decoupled. By Proposition 1.5, we can find $0 < w_1 \le \overline{w}_1$, such that (2.9) holds for $n = 1$. (Note that for H_1 as well as for H_2 the rooms R_k, $k \ge 3$, are decoupled.) Now suppose that $0 < w_j \le \overline{w}_j$, $j = 1, \ldots, n-1$, have already been found. We then employ Proposition 1.5 to join Ω_n, P_n and R_{n+1} together: Again, since the rooms R_k, $k > n$, are decoupled for H_{n-1} as well as for H_n, in order to control $(H_n + 1)^{-1} - (H_{n+1} + 1)^{-1}$ it is enough to estimate

$$(-\Delta_N^{\Omega_n} \oplus -\Delta_{DN}^{P_n(w_n)} \oplus -\Delta_N^{R_{n+1}} + 1)^{-1} - (-\Delta_N^{\Omega_{n+1}} + 1)^{-1}$$

and therefore Proposition 1.5 provides us with a $0 < w_n' \leq \overline{w}_n$ such that (2.9) holds for $0 < w_n \leq w_n'$. Applying also Proposition 1.6 to Ω_n, we can find $0 < w_n \leq w_n'$ such that

$$\left\| (-\Delta_N^{\Omega_n} + 1)^{-1} - (-\Delta_{DN}^{\Omega_n} + 1)^{-1} \right\| \leq \tfrac{1}{n^2}$$

(B) Now we fix a sequence $\{w_n\}$ which meets all the above requirements. Clearly, the form domains $\mathcal{Q}(\tilde{H}_n) = \mathcal{D}(\tilde{H}_n^{1/2})$ satisfy

$$\mathcal{Q}(\tilde{H}_n) \subset \mathcal{Q}(\tilde{H}_{n+1}) \subset \mathcal{H}^1(\Omega) = \mathcal{Q}(-\Delta_N^{\Omega})$$

for all $n \in \mathbf{N}$, and, by Lemma 1.7, they exhaust $\mathcal{H}^1(\Omega)$ in the sense that

$$\cup_{n=N}^{\infty} \mathcal{Q}(\tilde{H}_n) \text{ is dense in } \mathcal{H}^1(\Omega)$$

for all $N \in \mathbf{N}$. Since these quadratic forms are given by $\|\nabla u\|^2$, for $u \in \mathcal{Q}(\tilde{H}_n)$ or $u \in \mathcal{Q}(-\Delta_N^{\Omega})$, we may conclude that $\tilde{H}_n \to -\Delta_N^{\Omega}$ in strong resolvent sense, by standard convergence theorems for quadratic forms (cf. e.g. Kato [12; Theorem VIII–3.6 or Theorem VIII–3.11]).

Combining this result with (2.9) and (2.10), we see that $H_n \to -\Delta_N^{\Omega}$ in norm resolvent sense, and that

$$(H_1 + 1)^{-2} - (-\Delta_N^{\Omega} + 1)^{-2} \in \mathcal{B}_1$$

so that $\sigma_{\text{ac}}(-\Delta_N^{\Omega}) = \sigma_{\text{ac}}(H_1)$ by Kato–Birman theory (see e.g. Reed and Simon [14; pg. 30, Corollary 3]) and $\sigma_{\text{ess}}(-\Delta_N^{\Omega}) = \sigma_{\text{ess}}(H_1)$ by a theorem of Weyl and the spectral mapping theorem. Now, since H_1 is the fully decoupled operator, with Neumann boundary conditions in the rooms and Dirichlet–Neumann boundary conditions in the passages, it is clear that $\sigma_{\text{ac}}(H_1) = \emptyset$ and

$$\sigma_{\text{ess}}(H_1) = \sigma_{\text{ess}}(\bigoplus_{k=1}^{\infty} -\Delta_N^{R_k})$$

(Note that the operators $-\Delta_{DN}^{P_k}$ cannot contribute to $\sigma_{\text{ess}}(H_1)$ since the bottom of their spectrum goes to ∞, as $k \to \infty$.) This completes the proof of Theorem 2.1. QED

We now consider more specific R_k and begin with classical rooms and passages where each R_k is a rectangle. Let

$$R_k = (x_k\,, x'_k) \times (-\tfrac{1}{2}\eta_k\,, \tfrac{1}{2}\eta_k) \tag{2.12}$$

for some bounded sequence $\{\eta_k\}$, $\eta_k > 0$. As $|x_k - x'_k| \to 0$, $k \to \infty$, it is easy to see that

$$\sigma_{\text{ess}}(\bigoplus_{k=1}^{\infty} -\Delta_N^{R_k}) = \{0\} \cup \big\{m^2\pi^2\alpha \mid m \in \mathbf{N}\,,\ \alpha \in \Sigma\big\}$$

where

$$\Sigma = \big\{\text{limit points of } \{\eta_k^{-2}\}\big\} \tag{2.13}$$

and we obtain the following result:

Corollary 2.2. *Suppose the rooms R_k are given by (2.12), with $\{\eta_k\}$ a bounded sequence of positive numbers. Let Σ be as in (2.13). Then, there exists a sequence of widths $\{w_k\}$, $w_k \to 0$, such that the Neumann Laplacian on $\Omega = \Omega(\{w_k\})$ satisfies*

$$\sigma_{\text{ess}}(-\Delta_N^{\Omega}) = \{0\} \cup \pi^2 \bigcup_{m=1}^{\infty} m^2\Sigma.$$

Remark. By Theorem A.1 in the Appendix, Corollary 2.2 can be generalized to hold for all sequences $\{w_k\}$ which go to zero fast enough. We believe that the result holds true if $w_k \to 0$ at some exponential rate while the other quantities behave polynomially.

Corollary 2.2 determines the essential spectrum of the Neumann Laplacian on typical rooms and passages (for very narrow passages). However, due to the somewhat special structure of the set $\cup_{m=1}^{\infty} m^2\Sigma$, it does not provide a really satisfactory answer to the inverse problem (IP). While the best answer to (IP) will only be obtained by the construction of modified combs in Section 3, we shall now make some progress by replacing each room R_k by a small square room with a partition leaving open a "door", as shown in Figure 5.

Insert Figure 5 here.

These "double rooms" $\tilde{R}_k$ of side length k^{-2} will be chosen in such a way that $-\Delta_N^{\tilde{R}_k}$ has an eigenvalue 0, one low–lying eigenvalue less than $\pi^2 k^4$ (which can be adjusted by choosing the width a_k of the "door"), while the remaining eigenvalues are larger than $\pi^2 k^4$. In fact, we have the following lemma.

Lemma 2.3. *For $l > 0$ and $0 \leq \rho \leq l$, consider the open set in $\mathbf{R}^2$*

$$Q(l;\rho) = \big((-l,0) \cup (0,l)\big) \times (-l,l) \cup \{0\} \times (-\rho,\rho)$$

Then, for any $\mu \in (0,\pi^2/4l^2)$, there exists $\rho \in (0,l)$ such that the (repeated) eigenvalues $\lambda_j(\rho)$, $j = 0,1,2,\ldots$, of the Neumann Laplacian on $Q(l;\rho)$ satisfy

$$\lambda_0(\rho) = 0 \qquad \lambda_1(\rho) = \mu \qquad \lambda_j(\rho) \geq \frac{\pi^2}{4l^2} \quad (j \geq 2)$$

The proof of this lemma will be given at the end of this section. In the construction of rooms and passages, let us now assume that $|x_k' - x_k| = k^{-2}$ and that each room R_k is replaced by $\tilde{R}_k$, where $\tilde{R}_k$ is a square with a partition, leaving open a door of width a_k, as shown in Figure 4. We then have:

Corollary 2.4 (Modified Rooms and Passages). *Let $\lambda_1(a_k)$ denote the first non-zero eigenvalue of $-\Delta_N^{\tilde{R}_k}$. Then, for suitably chosen widths w_k of the passages P_k, the Neumann Laplacian on $\Omega = \Omega(\{w_k\})$ satisfies*

$$\sigma_{\mathrm{ess}}(-\Delta_N^{\Omega}) = \{0\} \cup \{\text{accumulation points of } \{\lambda_1(a_k)\}\}$$

An immediate consequence is the following inverse result:

Corollary 2.5. *For any closed set $S \subset [0, \infty)$, there exists an open, bounded, connected set $\Omega \subset \mathbf{R}^2$ such that*

$$\sigma_{\text{ess}}(-\Delta_N^\Omega) = \{0\} \cup S$$

In Section 3 we will construct examples which do not necessarily have 0 in the essential spectrum.

Proof of Lemma 2.3 We first exploit the monotonicity of the Sobolev spaces $\mathcal{H}^1\big(Q(l;\rho)\big)$ with respect to ρ,

$$\mathcal{H}^1\big(Q(l;\rho')\big) \subset \mathcal{H}^1\big(Q(l;\rho)\big) \qquad 0 \leq \rho \leq \rho' \leq l$$

to conclude that $\lambda_j(\rho) \leq \lambda_j(\rho')$, for $j = 0, 1, 2, \ldots$ and $0 \leq \rho \leq \rho' \leq l$.

For $\rho = 0$, we clearly have

$$\lambda_0(0) = \lambda_1(0) = 0 \qquad \lambda_2(0) = \frac{\pi^2}{4l^2}$$

so, by monotonicity,

$$\lambda_2(\rho) \geq \lambda_2(0) = \frac{\pi^2}{4l^2} \qquad 0 \leq \rho \leq l$$

Since $\lambda_1(0) = 0$ and $\lambda_1(l) = \pi^2/4l^2$, the result will follow if we can show that $\lambda_1(\rho)$ depends continuously on $\rho \in [0, l]$.

To prove continuity at 0, we choose a function $\psi \in C^\infty(\mathbf{R}^2)$, satisfying

$$\begin{aligned} \psi(x,y) &= 1 \qquad x^2 + y^2 > 4 \\ \psi(0,y) &= 0 \qquad -1 \leq y \leq 1 \end{aligned}$$

and let

$$\psi_\rho(x,y) = \psi(\rho^{-1}x, \rho^{-1}y) \qquad \rho > 0$$

Letting χ_R and χ_L denote the characteristic functions of the right and left portion of $Q(l;0)$ respectively, we define

$$\tilde{u}_\rho = \psi_\rho \cdot (\chi_R - \chi_L)$$

We have: $\tilde{u}_\rho$ is orthogonal to the constant function, $\tilde{u}_\rho \in \mathcal{H}^1(Q(l;\rho))$, $\|\nabla \tilde{u}_\rho\| \leq$ const and $\tilde{u}_\rho \to \chi_R - \chi_L$ in L^2, as $\rho \to 0$. Thus, for a suitable sequence $\{\rho_j\}$ converging to 0, we have $\tilde{u}_\rho \to \chi_R - \chi_L$ weakly in $\mathcal{H}^1\big(Q(l;0)\big)$, and the Banach–Saks theorem yields that the averages

$$v_N = \frac{1}{N}\sum_{j=1}^{N} \tilde{u}_{\rho_j}$$

converge to $\chi_R - \chi_L$, strongly in $\mathcal{H}^1\big(Q(l;0)\big)$. Therefore, given $\epsilon > 0$, we can find a function w_ϵ of norm 1, $w_\epsilon \in \mathcal{H}^1\big(Q(l;\rho)\big)$, for small ρ, satisfying

$$\|\nabla w_\epsilon\| < \epsilon \qquad \int_{Q(l;\rho)} w_\epsilon = 0$$

This proves $\lambda_1(\rho) < \epsilon$ for ρ sufficiently small.

Continuity of $\lambda_1(\rho)$ in $0 < \rho \leq l$ follows by monotonicity and a simple dilation argument. QED

3. Combs.

We now apply the techniques of Sections 1 and 2 to Neumann Laplacians on comb–like domains. Our combs are constructed by attaching an infinite number of thin "teeth" (rectangles) of finite length to a fixed square forming the basis of the comb; each tooth plays the role of one room and one passage simultaneously. In the second part of this section, we shall produce combs with more sophisticated teeth (teeth of shrinking size with partitions, similar to the double rooms in Section 2), which provide a complete answer to the inverse problem (IP) of Section 2. Each of the teeth with partitions will contribute to the spectrum of the decoupled comparison operator precisely one low–lying eigenvalue which again can be adjusted by choosing the opening of the "door", while the remaining eigenvalues will be very large.

We first describe ordinary combs.

Let the basis (or the "handle") of the comb be the set $\Omega_0 = (0,1) \times (-1,0) \subset \mathbf{R}^2$ and suppose we are given a bounded sequence $\{\eta_k\}$ of positive numbers. The η_k give the lenght of the k^{th} tooth, $k = 1, 2, \ldots$. The width w_k of the k^{th} tooth will be determined inductively.
Suppose $\{w_k\}$ is some sequence of positive numbers such that $\sum w_k < 1$. We then denote the initial x–coordinate of the k-th tooth by

$$a_k := \sum_{j=1}^{k-1} w_j \qquad k = 1, 2, \ldots \tag{3.1}$$

and the total width occupied by teeth as

$$A = \sum_{k=1}^{\infty} w_k \tag{3.2}$$

For the k-th tooth, let

$$D_k = (a_k, a_k + w_k) \times (0, \eta_k)\ , \quad D'_k = (a_k, a_k + w_k) \times [0, \eta_k) \tag{3.3}$$

The comb-domain is then given by

$$\Omega = \Omega_0 \cup (\cup_{k=1}^{\infty} D'_k)\ , \tag{3.4}$$

while the approximating comb with only the first n teeth left, is given by

$$\Omega_n = \Omega_0 \cup (\cup_{k=1}^{n} D'_k) \tag{3.5}$$

As in the rooms and passages example, we'll also need two kinds of approximating operators,

$$H_n = -\Delta_N^{\Omega_n} \oplus \left(\bigoplus_{k>n} -\Delta_{DN}^{D_k} \right) \quad n \in \mathbf{N}_0 \tag{3.6}$$

and

$$\tilde{H}_n = -\Delta_{DN}^{\Omega_n} \oplus \left(\bigoplus_{k>n} -\Delta_{DN}^{D_k} \right) \qquad n \in \mathbf{N}_0 \tag{3.7}$$

where the boundary conditions are chosen in the following way: $-\Delta_{DN}^{D_k}$ has Dirichlet boundary conditions on the line segment $[a_k, a_k + w_k] \times \{0\}$ and Neumann boundary conditions on the rest of ∂D_k; $-\Delta_{DN}^{\Omega_n}$ has Dirichlet conditions on the line segment $[a_{n+1}, A] \times \{0\}$, and Neumann boundary conditions on the remaining portions of $\partial\Omega_n$ (cf. Figure 6 below). In particular, all the teeth are decoupled from the basis Ω_0 for the operator H_0. Similarly, for $w_1, \ldots, w_n$ given, $-\Delta_{D_\delta N}^{\Omega_n}$ will denote the Laplacian on Ω_n with Dirichlet boundary conditions on the line segment $[a_{n+1}, a_{n+1} + \delta] \times \{0\}$, for $0 < \delta < 1 - a_{n+1}$ and Neumann boundary conditions on the rest of $\partial\Omega_n$. Note that, for fixed $\{\eta_k\}$, the domains and operators defined above will depend on the sequence $\{w_n\}$.

Insert Figure 6 here.

Proposition 3.1. *Suppose $w_1, \ldots, w_n$ are given, with $w_k > 0$, for $k = 1, \ldots, n$, and $\sum_{j=1}^n w_j < 1$. Then there exists $\tilde{w}_{n+1} > 0$ such that*

$$\left\| (H_n + 1)^{-1} - (H_{n+1} + 1)^{-1} \right\| \leq 1/(n+1)^2 \qquad n \geq 0 \tag{3.8}$$

$$\left\| (H_n + 1)^{-2} - (H_{n+1} + 1)^{-2} \right\|_{\mathcal{B}_1} \leq 1/(n+1)^2 \qquad n \geq 0 \tag{3.9}$$

provided $w_{n+1} \leq \tilde{w}_{n+1}$, for any choice of $w_{n+2}, w_{n+3}, \ldots$, but still assuming $\sum w_k < 1$.

Proof. Since the teeth $D_{n+2}, D_{n+3}, \ldots$ are decoupled for H_n as well as for H_{n+1}, it is enough to compare $(-\Delta_N^{\Omega_n} \oplus -\Delta_{DN}^{D_{n+1}} + 1)^{-1}$ and $(-\Delta_N^{\Omega_{n+1}} + 1)^{-1}$, and the result will follow if we can show that the assumptions of Proposition 1.9 are satisfied: Clearly, the small eigenvalues on the dents are independent of w, for w small, while Neumann bracketing yields that $(-\Delta_N^{\Omega_n} + 1)^{-2}$ is trace class. QED

We also have to consider the difference between the resolvent of H_n and $\tilde{H}_n$.

Proposition 3.2. *Let $n \geq 1$ and suppose $w_1, \ldots, w_n$ are given, with $w_k > 0$, for $k = 1, \ldots, n$, and $\sum w_k < 1$. Then, there exists $\delta_n > 0$, such that*

$$\left\|(H_n + 1)^{-1} - (\tilde{H}_n + 1)^{-1}\right\| \leq \frac{1}{n^2} \tag{3.10}$$

for any choice of $w_{n+1}, w_{n+2}, \ldots$, provided $\sum_{k>n} w_k < \delta_n$.

Proof. Since the teeth $D_{n+1}, D_{n+2}, \ldots$ are decoupled for H_n as well as for $\tilde{H}_n$, it is clearly enough to ensure the existence of a δ_n such that

$$\left\|(-\Delta_N^{\Omega_n} + 1)^{-1} - (-\Delta_{D_\delta N}^{\Omega_n} + 1)^{-1}\right\| \leq \frac{1}{n^2} \tag{3.11}$$

for all $0 < \delta \leq \delta_n$ (recall the definition of $-\Delta_{D_\delta N}^{\Omega_n}$, given at the beginning of this section). Hence the desired result follows from Proposition 1.6.

QED

Again, the form domains $\mathcal{Q}(\tilde{H}_n)$ are non-decreasing and they exhaust $\mathcal{H}^1(\Omega)$, the form domain of $-\Delta_N^\Omega$, in the following sense:

Proposition 3.3. *Let $\{w_k\}$ be a sequence of positive numbers with $\sum w_k < 1$. Then $\cup_{n=N}^\infty \mathcal{Q}(\tilde{H}_n)$ is dense in $\mathcal{H}^1(\Omega)$, for $N \in \mathbf{N}$.*

Proof. Let $u \in \mathcal{H}^1(\Omega)$. Lemma 1.7 provides a sequence $\{u_k\} \subset \mathcal{H}^1(\Omega)$ such that $\|u - u_k\|_1 \to 0$ and $u_k(x, y) = 0$ for (x, y) in the ball of radius $1/k$, centerd at the point $(A, 0)$. Hence $u_k \in \mathcal{Q}(\tilde{H}_n)$, for n sufficiently large, and the result follows. QED

We are now ready to put the pieces together.

Theorem 3.4. *Suppose we are given a bounded sequence of lengths $\{\eta_k\}$, $\eta_k > 0$. Then there exists a sequence $\{w_k\}$ of widths, $w_k > 0$, such that the Neumann Laplacian on the domain $\Omega = \Omega(\{\eta_k\}, \{w_k\})$, defined as in (3.1)–(3.4), enjoys the following properties:*

(i) $-\Delta_N^\Omega$ *has no absolutely continuous spectrum.*

(ii) $\sigma_{\text{ess}}(-\Delta_N^\Omega) = \pi^2 \cup_{m=0}^{\infty} (\frac{2m+1}{2})^2 \Sigma$, *where Σ is the set of limit points of the sequence $\{\eta_k^{-2}\}$.*

Remark. Fleckinger and Métivier [9] consider a class of combs with *compact* $(-\Delta_N + 1)^{-1}$, where they derive results on the asymptotic distribution of eigenvalues.

Proof. By Proposition 3.1, we can find some $w_1 > 0$ such that (3.8) and (3.9) hold. Using Proposition 3.1 and 3.2 we then choose $w_k > 0$ inductively, making sure that (as we pass from k to $k+1$)

(i) $w_{k+1} < \tilde{w}_{k+1}$ with $\tilde{w}_{k+1}$ as in Proposition 3.1.

(ii) $\sum_{j=1}^{k+1} w_j < \sum_{j=1}^{s} w_j + \delta_s$ for $s = 1, \ldots, k$,

whith δ_s as in Proposition 3.2. (The meaning of condition *(ii)* is the following: If $w_1, \ldots, w_s$ have been defined, then Proposition 3.2 imposes the restriction $\sum_{k>s} w_k < \delta_s$, and this is for $s = 1, 2, \ldots$.) Hence, for this sequence $\{w_k\}$, (3.8) and (3.9) hold for all $n \in \mathbf{N}$.

By Proposition 3.3 and Kato [12 ; Theorem VIII–3.6 or Theorem 3.11], we are able to conclude that $\tilde{H}_n \to -\Delta_N^\Omega$ in strong resolvent sense. It follows by (3.8) and (3.9) that $H_n \to -\Delta_N^\Omega$ in norm resolvent sense, and that

$$(H_0 + 1)^{-2} - (-\Delta_N^\Omega + 1)^{-2} \in \mathcal{B}_1$$

so that $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0)$ and $\sigma_{\text{ac}}(H) = \sigma_{\text{ac}}(H_0)$.

Again, it is evident that $\sigma_{\mathrm{ac}}(H_0) = \emptyset$ and

$$\begin{aligned}\sigma_{\mathrm{ess}} &= \sigma_{\mathrm{ess}}\left(\bigoplus_{k=1}^{\infty} -\Delta_{DN}^{D_k}\right) \\ &= \left\{((2m+1)/2)^2\pi^2\alpha \mid m \in \mathbf{N}_0,\ \alpha \in \Sigma\right\}\end{aligned}$$

and we are done. QED

In order to arrive at a full solution of our inverse problem (IP), we now modify the comb construction, using teeth of shrinking size with partitions of the following precise type:

For $w > 0$ and $0 \le \gamma \le w$, the tooth $D(w,\gamma)$ is a rectangle of height $w + w^{1/4}$ and width w, with a horizontal partition at height $w^{1/4}$ which leaves open a door of width γ (cf. Figure 7). In the actual construction, we'll attach a sequence of such teeth to $\Omega_0 = (0,1) \times (-1,0)$. Again, let $-\Delta_{DN}^{D(w,\gamma)}$ denote the Laplacian on $D(w,\gamma)$ with Dirichlet boundary conditions on the bottom and Neumann boundary conditions on the remaining parts of the boundary. We will use the following analogue of Lemma 2.3 to determine the parameter γ in the further construction.

Insert Figure 7 here.

Lemma 3.5. *Let $0 \le \mu_1 \le \cdots \le \mu_i \le \cdots$, $\mu_i = \mu_i(w,\gamma)$, denote the (repeated) eigenvalues of $-\Delta_{DN}^{D(w,\gamma)}$, for $0 < w \le 1$ and $0 \le \gamma \le w$. We have:*

(a) $\mu_i(w,\gamma) \ge \mu_i(w,0)$, for $i \in \mathbf{N}$. In particular, $\mu_2 \to \infty$ as $w \to 0$, and, more strongly, $\sum_{i\ge 2}(\mu_i+1)^{-2} \to 0$ as $w \to 0$, uniformly in $0 \le \gamma \le w$.

(b) Suppose $\lambda > 0$ is given, and $w_\lambda > 0$ is such that $\mu_2(w,0) > \lambda$, for $0 < w < w_\lambda$. Then, for any $0 < w \le w_\lambda$, there exists a $\gamma = \gamma(w,\lambda) \in (0,w)$ such that

$$\mu_1\big(w, \gamma(w,\lambda)\big) = \lambda \qquad 0 < w \le w_\lambda$$

The proof is similar to the proof of Lemma 2.3.

Suppose now we are given a sequence $\{\lambda_k\} \subset (0,\infty)$. The preceeding Lemma enables us to find $\bar{w}_k > 0$ and functions $\gamma_k(w)$, defined for $0 < w \leq \bar{w}_k$, such that the first eigenvalue of the Dirichlet–Neumann Laplacian on $D\big(w, \gamma_k(w)\big)$ is just λ_k, whereas the second eigenvalue is greater than k. We denote these families of teeth by $D_k(w)$, $0 < w \leq \bar{w}_k$. (In fact, in the actual construction of combs, we will use translates of these D_k, but we will not make this explicit in the notation.)

Next, we define the objects a_k, Ω_n, Ω, H_n and $\tilde{H}_n$ as in (3.1)–(3.7), with the only difference that each tooth is now a set $D_k(w)$, translated in the x–direction by an amount of a_k. Clearly, statements and proofs of Propositions 3.2 and 3.3 apply essentially unchanged. By Lemma 3.5(a), and since $\mu_1(-\Delta_{DN}^{D_k(w)}) = \lambda_k$, by construction, we are in the position to apply Proposition 1.9 (also note that, by Neumann bracketing, $(-\Delta_N^{\Omega_n} + 1)^{-2}$ is trace class, for all n), and we obtain the estimates (3.8) and (3.9) of Proposition 3.1. This leads to the following main result.

Theorem 3.6. *Suppose we are given a sequence $\{\lambda_k\} \subset (0,\infty)$. Then, there exists a bounded, open, connected set $\Omega \subset \mathbf{R}^2$ such that $\sigma_{\text{ac}}(-\Delta_N^\Omega) = \emptyset$ and*

$$\sigma_{\text{ess}}(-\Delta_N^\Omega) = \{ \textit{ limit points of } \{\lambda_k\}\}$$

Proof. Again, we use (analogues of) Propositions 3.1–3.3 in order to find a sequence $\{w_k\}$ of positive numbers so that all the estimates (3.8)–(3.10) hold.
As before, $\tilde{H}_n \geq \tilde{H}_{n+1} \geq \cdots \geq 0$, and so Proposition 3.3 combined with the usual convergence theorems for quadratic forms implies that $\tilde{H}_n \to -\Delta_N^\Omega$ in strong resolvent sense. By (3.8)–(3.10) this yields $H_n \to -\Delta_N^\Omega$ in norm resolvent sense, and

$$(H_0 + 1)^{-2} - (-\Delta_N^\Omega + 1)^{-2} \in \mathcal{B}_1$$

Hence, $\sigma_{\text{ac}}(-\Delta_N^\Omega) = \sigma_{\text{ac}}(H_0) = \emptyset$ and

$$\begin{aligned} \sigma_{\text{ess}}(-\Delta_N^\Omega) &= \sigma_{\text{ess}}\Big(\bigoplus_{k=1}^{\infty} -\Delta_{DN}^{D_k(w_k)}\Big) \\ &= \{ \text{ limit points of } \{\lambda_k\}\} \end{aligned}$$

and we are done. QED

It is clear that we can construct Ω as small as we please, without changing the result of Theorem 3.6. This leads to the following solution of the inverse problem (IP).

Corollary 3.7. *For any closed set $S \subset [0,\infty)$, there exists a bounded, connected set Ω contained in the unit ball of $\mathbf{R}^2$, such that*

$$\sigma_{\text{ess}}(-\Delta_N^\Omega) = S.$$

Appendix

In this appendix, we discuss the main modifications needed in Propositions 1.4 and 1.6 to derive the following stronger version of Theorem 2.1.

Theorem A.1. *Suppose we are given a sequence of open domains $R_k \subset \mathbf{R}^2$, $k = 1, 2, \ldots$, which satisfy conditions (2.1)–(2.3). Then there exists a sequence $\{\overline{w}_k\}$ of positive numbers such that*

$$\sigma_{\text{ac}}(-\Delta_N^\Omega) = \emptyset, \quad \sigma_{\text{ess}}(-\Delta_N^\Omega) = \sigma_{\text{ess}}(\bigoplus_{k=1}^{\infty} -\Delta_N^{R_k})$$

for any sequence $\{w_k\}$ satisfying $0 < w_k \leq \overline{w}_k$, $k \in \mathbf{N}$, where $\Omega = \Omega(\{w_k\})$ is given by (2.5).

We shall need the estimates provided in Propositions 1.5 and 1.6 in a form which is largely independent of the domains involved, in the sense that w_ϵ can be chosen simultaneously for a whole family of domains $\Omega_q^{(t)}$, $0 \leq t \leq 1$, $q = 1, 2$. In the sequel, let B_ϱ denote the ball of radius ϱ, centered at the origin in $\mathbf{R}^2$, for $\rho > 0$.

Lemma A.2. *Consider a family of domains $\Omega^{(t)}$, $0 \le t \le 1$, $\Omega^{(t)}$ contained in the left half-plane in $\mathbf{R}^2$, satisfying*

$$(-s_0, 0) \times (-s_0, s_0) \subset \Omega^{(t)} \qquad 0 \le t \le 1$$

for some $s_0 > 0$. Then, for any $M > 0$ there exists a $C > 0$ such that

$$\sup \left|\psi \lceil B_{s_0/2}\right| + \sup \left|\nabla\psi \lceil B_{s_0/2}\right| \le C$$

for any normalized eigenfunction ψ of $-\Delta_N^{\Omega^{(t)}}$ associated with an eigenvalue $\lambda \le M$.

Proof. Let $t \in [0,1]$, $\lambda \le M$ and suppose $\psi \in \mathcal{D}(-\Delta_N^{\Omega^{(t)}})$ satisfies $\|\psi\| = 1$ and $-\Delta_N^{\Omega^{(t)}}\psi = \lambda\psi$. Reflection along the y-axis yields a function $\tilde{\psi} \in \mathcal{H}^1(B_s)$, which is a weak solution of $-\Delta_N^{\Omega^{(t)}}\tilde{\psi} = \lambda\tilde{\psi}$ in B_s, for $0 < s < s_0$. The desired result then follows by repeated use of the a-priori-estimates given in Gilbarg and Trudinger [10; Theorem 8.10], and an application of the Sobolev embedding theorem. QED

We now join two *families* of domains $\Omega_1^{(t)}$ and $\Omega_2^{(t)}$ by a narrow passage $P'_w = [0, L] \times (-w, w)$. In view of Lemma A.2 we require $\Omega_1^{(t)}$ to be of the type described above, while the domains $\Omega_2^{(t)}$ should lie to the right of $\{L\} \times \mathbf{R}$ and should contain the set $(L, L + s_0) \times (-s_0, s_0)$. We furthermore require the operators $(-\Delta_N^{\Omega_q^{(t)}} + 1)^{-1}$ to be compact, for $q = 1, 2$ and $0 \le t \le 1$. Again, let $\lambda_i^{(t)}$ and $\tilde{\lambda}_i^{(t)}$ denote the (repeated) eigenvalues of

$$H_w^{(t)} = -\Delta_N^{\Omega^{(t)}}$$

where

$$\Omega^{(t)} = \Omega_1^{(t)} \cup P'_w \cup \Omega_2^{(t)}$$

and of the decoupled operator

$$\tilde{H}_w^{(t)} = -\Delta_N^{\Omega_1^{(t)}} \oplus -\Delta_{DN}^{P_w} \oplus -\Delta_N^{\Omega_2^{(t)}}$$

respectively, where the mixed boundary conditions on the passage are chosen as in Section 1. Let $\{E_\lambda^{(t)}\}$, $\{\tilde{E}_\lambda^{(t)}\}$ denote the spectral families associated with $H_w^{(t)}$ and $\tilde{H}_w^{(t)}$, respectively. We have

Proposition A.3. *In addition to the assumptions made above, suppose that for any $\Lambda > 0$ there exists a constant C_Λ such that*

$$\sharp\{i \mid \tilde{\lambda}_i^{(t)} \leq \Lambda\} \leq C_\Lambda \qquad 0 \leq t \leq 1.$$

Then, for any $\epsilon > 0$, there exists $w_\epsilon > 0$ such that

$$\left\| E_\lambda^{(t)} - \tilde{E}_\lambda^{(t)} \right\| < \epsilon \qquad 0 < w < w_\epsilon$$

for all $\lambda \leq \Lambda$ which satisfy $\mathrm{dist}(\lambda, \sigma(\tilde{H}_w^{(t)})) > \epsilon$.

Proof. Using Lemma A.2, we are in a position to control the extension process $\tilde{\varphi}_i \mapsto \tilde{\psi}_i$, described just before Proposition 1.4, in a t-independent way: We obtain

$$\left\| \tilde{\psi}_i \lceil P_w \right\|_1 \to 0, \qquad w \to 0$$

uniformly in t. The rest of the proof is similar to the proof of Proposition 1.4. QED

It is now easy to obtain the following generalization of Proposition 1.5:

Proposition A.4. *Let $\Omega_q^{(t)}$, $q = 1, 2$, $0 \leq t \leq 1$ be as above, and suppose that, in addition,*

$$\sum_{i > N} (\tilde{\lambda}_i^{(t)} + 1)^{-2} \to 0, \qquad N \to \infty$$

uniformly in $0 \leq t \leq 1$. Then

$$\left\| (H_w^{(t)} + 1)^{-1} - (\tilde{H}_w^{(t)} + 1)^{-1} \right\| \to 0, \quad w \to 0,$$

$$\left\| (H_w^{(t)} + 1)^{-2} - (\tilde{H}_w^{(t)} + 1)^{-2} \right\|_{\mathcal{B}_1} \to 0, \quad w \to 0,$$

uniformly in $0 \leq t \leq 1$.

Proof. Only some obvious changes in the proof of Proposition 1.5. QED

We finally have to change Neumann boundary conditions on the line segment $\{0\} \times [-w,w]$ into a Dirichlet boundary condition.

Proposition A.5. *Let $\Omega^{(t)}$ be as in Lemma A.2, and let $-\Delta^{\Omega^{(t)}}_{D_w N}$ obey Dirichlet boundary condition on $\{0\} \times [-w,w]$, and Neumann boundary condition on the remaining portions of $\partial\Omega^{(t)}$. Also assume that*

$$\lambda_i^{(t)} \to \infty, \qquad i \to \infty,$$

uniformly in $0 \le t \le 1$, where the $\lambda_i^{(t)}$ denote the (repeated) eigenvalues of $-\Delta_N^{\Omega^{(t)}}$. Then

$$\left\| (-\Delta_N^{\Omega^{(t)}} + 1)^{-1} - (-\Delta^{\Omega^{(t)}}_{D_w N} + 1)^{-1} \right\| \to 0, \qquad w \to 0,$$

uniformly in $0 \le t \le 1$.

Proof. The proof is similar to the proof of Proposition 1.6 but it requires a t-independent version of Lemma 1.7 for eigenfunctions; see Lemma A.6 below. QED

Lemma A.6. *Let $\Omega^{(t)}$ be as above, and let $\Lambda > 0$. Then there exists a sequence of cut-off functions $\phi_k \in C^\infty(\mathbf{R}^2)$ with the following properties:*

(a) $0 \le \phi_k \le 1$,

(b) ϕ_k vanishes in a neighborhood of the origin,

(c) for any $\epsilon > 0$, there exists $k_0 \in \mathbf{N}$ such that for $k > k_0$,

$$\|u - \phi_k u\|_{\mathcal{H}^1(\Omega^{(t)})} < \epsilon$$

for all normalized eigenfunctions u of $-\Delta_N^{\Omega^{(t)}}$, associated with eigenvalues smaller than Λ, for $0 \le t \le 1$.

Proof. Applying the Banach-Saks theorem to the sequence of cut-offs φ_k used in Lemma 1.7, we see that, for a suitable sequence $\{k_j\} \subset \mathbf{N}$

$$\left\| \frac{1}{N} \sum_{j=1}^{N} \varphi_{k_j} \lceil B_2 \right\|_{\mathcal{H}^1(B_2)} \to 0 \qquad N \to \infty$$

We then may define $\phi_N = 1/N \sum_{j=1}^{N} \varphi_{k_j}$, for $N \in \mathbf{N}$. As the eigenfunctions u (together with their gradients) obey a uniform bound on the ball $B_{s_0/2}$, by Lemma A.2, the result follows by straightforward estimates. QED

Proof of Theorem A.1. The proof now follows closely the lines of proof of Theorem 2.1, using Propositions A.4 and A.5 in place of Propositions 1.5 and 1.6, respectively. QED

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