The Myth of Hedge Fund Fee Diversification

Fei Meng*
David Saunders†
Luis Seco‡

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Abstract

Many innovative hedge fund fee structures have been introduced in recent years in response to concerns about both the level of hedge fund fees, and the incentives that they may provoke. A traditional fee structure consists of a flat fee charged as a percentage of the assets under management, together with a performance fee consisting of a percentage of the profits earned. A fee structure that has become more popular recently is the first-loss structure, in which the manager receives a higher performance fee in return for providing some downside protection to investors by insuring some of their losses. Combinations of these fee structures have also been proposed, with the assertion that investors may benefit from some diversification among the fee structures. By considering the investors' risk-reward tradeoff, we show that there is in fact very little benefit from such fee diversification.

*Department of Statistics and Actuarial Science, University of Waterloo, f7meng@uwaterloo.ca
†Department of Statistics and Actuarial Science, University of Waterloo, dsaunders@uwaterloo.ca
‡Department of Mathematics, University of Toronto, and Sigma Analysis and Management Ltd., seco@math.utoronto.ca

1 Introduction

Hedge funds feature specially designed fee structures to motivate managers. Typically, the fee structure of a hedge fund consists of two parts: a management fee and a performance fee (perhaps restricted by a high-water mark provision). The management fee is charged as a percentage of assets under management and the performance fee is charged as a percentage of the investor's profits. For instance, a traditional fee structure consists of a management fee of 2% of assets under management and a performance fee of 20% of net profits.

Hedge funds have faced intense scrutiny since the financial crisis, as the fees they charge investors have been outsized compared to the returns they are posting as a group.[1] Consequently, innovative fee structures have emerged aiming at better alignment between investors' interests and the hedge fund business objectives. An example is the class of first-loss fee structures (see Banzaca (2012) for a description). In these structures, in return for a higher performance fee, the hedge fund manager provides some downside protection to the investors on their losses. He and Kou (2016) present a description, a mathematical analysis of the incentives that such a fee structure provokes, and study the impact on the utility of both the hedge fund manager and investor, while Djerroud *et al.* (2016a) analyze the first-loss fee structure using an option-pricing perspective, leading to the identification of fair levels of the performance fee. Fee innovation has led to discussions and negotiations between investors and managers on the optimal fee to be used in certain situations, amplifying the universe of available fee structures by mixing different structures together. One example is the shared-loss structure, which can be considered as a mixture of the classical structure and the first-loss structure. Under a shared-loss agreement, rather than covering *all* investor losses up to a certain limit, the manager will provide compensation for a proportion of the investor's losses (again subject to a ceiling). To draw an (imperfect) analogy with insurance, in the first loss structure, the manager provides full insurance on losses (up to a preset limit), while in the shared loss structure the manager is providing partial insurance.

Perhaps owing to the randomness of both the fee payout and the fund returns to the investor (gross and net), a conventional wisdom has emerged on the investor side that *fee diversification*, or shared loss, is superior to both the traditional fee

[1] https://www.bloomberg.com/news/articles/2017-06-07/new-york-illinois-pension-funds-say-hedge-funds-fees-too-high

structures and the first-loss. The rationale for this conception borrows the concept of diversification from portfolio theory to conclude that the optimal fee structure should be a combination of the extremes. It is held that this diversification effect will not show up in the (risk-neutral) valuation of the fee structures, but rather when examining the risk-return tradeoff among fee structures.

In Djerroud *et al.* (2016b), the fairness of the prevailing levels of hedge fund fees were investigated in terms of risk-neutral valuation. In this paper we put the hypothesis of fee diversification to the test, by examining the problem of an investor who may choose any combination of the first loss and classical fee structures, and seeks to maximize the Sharpe ratio of their final payoff (evaluated using real-world probabilities). We demonstrate that for the vast majority of fund mean returns and volatilities, there is no fee diversification effect: either the first-loss structure or the classical structure is optimal for the investor. We also identify the (very small) region of the parameter space in which fee diversification prevails, and demonstrate that even in this region the effect of fee diversification is negligible.

The remainder of the paper is organized as follows. Section 2 describes our model and presents the optimal solution of the Sharpe Ratio maximization problem. Section 3 provides numerical results and Section 4 concludes the paper.

2 Sharpe Ratio Maximization with Portfolios of Fee Structures

Assume the hedge fund's dynamics under the real-world measure follow:

$$dX_t^x = \mu X_t^x dt + \sigma X_t^x dW_t, \quad X_0^x = x, \tag{2.1}$$

where μ is the growth rate and σ is the fund volatility. Next, let m_1, and α_1 denote the management fee and performance fee for the traditional fee structure, and m_2, and α_2 denote the management fee and performance fee for the first-loss fee structure. Moreover, we assume the management fees are proportional to the initial investment x and performance fees are proportional to the final hedge fund value X_T^x. Thus the

payoff function for the investor in the traditional fee structure at maturity is:

$$g_1(X_T^x) = \begin{cases} X_T^x - m_1 x - \alpha_1(X_T^x - m_1 x - x), & X_T^x \geq x + m_1 x, \\ X_T^x - m_1 x, & X_T^x < x + m_1 x, \end{cases} \tag{2.2}$$

while the payoff function for the first-loss fee structure at maturity is:

$$g_2(X_T^x) = \begin{cases} X_T^x - m_2 x - \alpha_2(X_T^x - m_2 x - x), & X_T^x \geq x + m_2 x, \\ x, & x + m_2 x - cx \leq X_T^x < x + m_2 x, \\ X_T^x - m_2 x + cx, & X_T^x < x + m_2 x - cx. \end{cases} \tag{2.3}$$

where c is the deposit amount, giving the fraction of the initial investment that is insured (see Djerroud *et al.* (2016b)). Now, let $\omega_1 \geq 0$ and $\omega_2 \geq 0$ denote the proportion of the fund invested in the traditional fee structure and first-loss structure respectively, where $\omega_1 + \omega_2 = 1$. Then, the investor's final payoff becomes

$$g(X_T^x) = g_1(\omega_1 X_T^x) + g_2(\omega_2 X_T^x) = \omega_1 g_1(X_T^x) + \omega_2 g_2(X_T^x). \tag{2.4}$$

where the second equation follows since $g_i(\omega_i X_T^x) = g_i(X_T^{\omega_i x}) = \omega_i g_i(X_T^x)$, for $i = 1, 2$, $\omega_i \geq 0$.

Assuming no early withdrawals, and an investment horizon of T, the investor's Sharpe ratio becomes:

$$S(T) := \frac{\mathbb{E}[r(T)] - r_{f,T}}{\sqrt{\mathrm{Var}[r(T)]}}, \tag{2.5}$$

where $r(T) = g(X_T^x)/x - 1$ is the fund investor's return, and $r_{f,T}$ is the risk-free interest rate. Using

$$\mathbb{E}[r(T)] = \frac{\mathbb{E}[g(X_T)]}{x} - 1, \quad \mathrm{Var}[r(T)] = \frac{\mathrm{Var}[g(X_T)]}{x^2}. \tag{2.6}$$

we obtain:

$$S(T) = \frac{\mathbb{E}[g(X_T)] - (1 + r_{f,T})x}{\sqrt{\mathrm{Var}[g(X_T)]}}. \tag{2.7}$$

Now, introduce the notation

$$\begin{aligned}
&\mu_1 = \mathbb{E}[g_1(X_T^x)], \quad \mu_2 = \mathbb{E}[g_2(X_T^x)],\\
&\sigma_1 = \mathrm{Var}[g_1(X_T^x)], \quad \sigma_2 = \mathrm{Var}[g_2(X_T^x)],\\
&\sigma_{12} = \mathrm{Cov}[g_1(X_T^x),\ g_2(X_T^x)].
\end{aligned}$$

And, from (2.4), we can write,

$$\mathbb{E}[g(X_T^x)] = \boldsymbol{\omega}^T\boldsymbol{\mu}, \quad \mathrm{Var}[g(X_T^x)] = \boldsymbol{\omega}^T\Sigma\boldsymbol{\omega}. \tag{2.8}$$

where

$$\boldsymbol{\omega} = \begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix}, \quad \Sigma = \begin{bmatrix}\sigma_1 & \sigma_{12}\\ \sigma_{12} & \sigma_2\end{bmatrix}. \tag{2.9}$$

Here, we assume that the investor seeks to maximize the Sharpe Ratio $S(T)$. As a result, the investor is solving the following optimization problem:

$$\begin{aligned}
&\max_{\boldsymbol{\omega}} \ \frac{\boldsymbol{\omega}^T\boldsymbol{\mu} - (1+r_{f,T})x}{\sqrt{\boldsymbol{\omega}^T\Sigma\boldsymbol{\omega}}}\\
&\text{subject to} \ \ \boldsymbol{\omega}^T\mathbf{1} = 1,\\
&\qquad\qquad \boldsymbol{\omega} \geq 0.
\end{aligned} \tag{2.10}$$

First, we can simplify (2.10) by considering the payoff functions:

$$\tilde{g}_1(X_T^x) = g_1(X_T^x) - (1+r_{f,T})x, \tag{2.11}$$

$$\tilde{g}_2(X_T^x) = g_2(X_T^x) - (1+r_{f,T})x. \tag{2.12}$$

Then, we can easily verify that

$$\begin{aligned}
&\tilde{\mu}_1 = \mathbb{E}[\tilde{g}_1(X_T^x)] = \mu_1 - (1+r_{f,T})x, \quad \tilde{\mu}_2 = \mathbb{E}[\tilde{g}_2(X_T^x)] = \mu_2 - (1+r_{f,T})x,\\
&\tilde{\sigma}_1^2 = \mathrm{Var}[\tilde{g}_1(X_T^x)] = \sigma_1^2, \quad \tilde{\sigma}_2^2 = \mathrm{Var}[\tilde{g}_2(X_T^x)] = \sigma_2^2, \quad \tilde{\sigma}_{12} = \mathrm{Cov}[\tilde{g}_1(X_T^x), \tilde{g}_2(X_T^x)] = \sigma_{12}.
\end{aligned}$$

So, the problem (2.10) can be expressed as follows,

$$\max_{\boldsymbol{\omega}} \ \frac{\boldsymbol{\omega}^T\tilde{\boldsymbol{\mu}}}{\sqrt{\boldsymbol{\omega}^T\tilde{\Sigma}\boldsymbol{\omega}}}$$

$$\begin{aligned} &\text{subject to } \boldsymbol{\omega}^T \mathbf{1} = 1, \\ &\boldsymbol{\omega} \geq 0. \end{aligned} \tag{2.13}$$

where

$$\tilde{\boldsymbol{\mu}} := \begin{bmatrix} \tilde{\mu}_1 \\ \tilde{\mu}_2 \end{bmatrix}, \quad \tilde{\Sigma} = \begin{bmatrix} \tilde{\sigma}_1 & \tilde{\sigma}_{12} \\ \tilde{\sigma}_{12} & \tilde{\sigma}_2 \end{bmatrix}. \tag{2.14}$$

Now, we can rewrite (2.13) as a standard quadratic programming problem if we assume $\tilde{\mu}_1 > 0$ and $\tilde{\mu}_2 > 0$. This is a natural assumption here because the investor anticipates a higher expected return than the risk-free rate when investing in the hedge fund. If we let $f(\boldsymbol{\omega})$ denote the objective function in (2.13), then it can be verified that for any real number $\lambda > 0$, $f(\boldsymbol{\omega}) = f(\lambda\boldsymbol{\omega})$. Thus, the problem (2.13) is equivalent to the optimization problem:

$$\begin{aligned} &\max_{\boldsymbol{\omega}} \frac{1}{\sqrt{\boldsymbol{\omega}^T \tilde{\Sigma} \boldsymbol{\omega}}} \\ &\text{subject to } \boldsymbol{\omega}^T \tilde{\boldsymbol{\mu}} = 1, \\ &\boldsymbol{\omega} \geq 0. \end{aligned} \tag{2.15}$$

Clearly, this can be rewritten as an equivalent minimization problem:

$$\begin{aligned} &\min_{\boldsymbol{\omega}} \boldsymbol{\omega}^T \tilde{\Sigma} \boldsymbol{\omega} \\ &\text{subject to } \boldsymbol{\omega}^T \tilde{\boldsymbol{\mu}} = 1, \\ &\boldsymbol{\omega} \geq 0. \end{aligned} \tag{2.16}$$

which is a standard quadratic programming problem.

Theorem 2.1. *Suppose $\tilde{\mu}_1 \geq 0$ and $\tilde{\mu}_2 \geq 0$. Let $\boldsymbol{\omega}^* = (\omega_1^*, \omega_2^*)$ be the optimal solution for* (2.16)*. If $\tilde{\sigma}_{12} \leq \min\{\frac{\tilde{\mu}_1}{\tilde{\mu}_2}\tilde{\sigma}_2^2, \ \frac{\tilde{\mu}_2}{\tilde{\mu}_1}\tilde{\sigma}_1^2\}$ is satisfied, then*

$$\omega_1^* = \frac{\tilde{\mu}_1\tilde{\sigma}_2^2 - \tilde{\mu}_2\tilde{\sigma}_{12}}{C} > 0, \quad \omega_2^* = \frac{\tilde{\mu}_2\tilde{\sigma}_1^2 - \tilde{\mu}_1\tilde{\sigma}_{12}}{C} > 0, \tag{2.17}$$

where $C = (\tilde{\mu}_1\tilde{\sigma}_2 - \tilde{\mu}_2\tilde{\sigma}_1)^2 + 2\tilde{\mu}_1\tilde{\mu}_2(\tilde{\sigma}_1\tilde{\sigma}_2 - \tilde{\sigma}_{12})$. *Otherwise,*

$$\boldsymbol{\omega}^* = \begin{cases} (1/\tilde{\mu}_1, 0), & \text{if } \frac{\tilde{\sigma}_1^2}{\tilde{\mu}_1^2} \leq \frac{\tilde{\sigma}_2^2}{\tilde{\mu}_2^2}, \\ (0, 1/\tilde{\mu}_2), & o.w. \end{cases} \tag{2.18}$$

Proof. By Best (2010, Chapter 9, Page 192), we can obtain the following optimality conditions for (2.16),

$$\begin{cases} \boldsymbol{\omega}^T\tilde{\boldsymbol{\mu}} = 1, \\ \mathbf{I}\lambda - \mathbf{2\Sigma\omega} = \nu\boldsymbol{\mu}, \\ \lambda^T\mathbf{I}\boldsymbol{\omega} = \mathbf{0}, \\ \boldsymbol{\omega} \geq 0, \\ \lambda \geq 0. \end{cases} \tag{2.19}$$

where ν is the multiplier for the constraint $\boldsymbol{\omega}^T\tilde{\boldsymbol{\mu}} = 1$ and λ is the vector of multipliers for the constraints $\boldsymbol{\omega} \geq 0$. More explicitly, this leads to the linear system:

$$\omega_1\tilde{\mu}_1 + \omega_2\tilde{\mu}_2 = 1, \tag{2.20}$$

$$\lambda_1 - 2\omega_1\tilde{\sigma}_1^2 - 2\omega_2\tilde{\sigma}_{12} = \nu\tilde{\mu}_1, \tag{2.21}$$

$$\lambda_2 - 2\omega_1\tilde{\sigma}_{12} - 2\omega_2\tilde{\sigma}_2^2 = \nu\tilde{\mu}_2, \tag{2.22}$$

$$\lambda_1\omega_1 + \lambda_2\omega_2 = 0. \tag{2.23}$$

with constraints $\boldsymbol{\omega} \geq 0$ and $\lambda \geq 0$. The constraints imply the solution to (2.23) must satisfy one of the following three cases: (i) $\lambda_1 = 0$ and $\lambda_2 = 0$. (ii) $\lambda_1 > 0$ and $\lambda_2 = 0$. (iii) $\lambda_1 = 0$ and $\lambda_2 > 0$.

It is easy to see check that case (i) can be viewed as an optimization problem:

$$\begin{aligned} &\min_{\boldsymbol{\omega}} \ \boldsymbol{\omega}^T\tilde{\Sigma}\boldsymbol{\omega} \\ &\text{Subject to } \boldsymbol{\omega}^T\tilde{\boldsymbol{\mu}} = 1. \end{aligned} \tag{2.24}$$

Let $\boldsymbol{\omega}^{**} = (w_1^{**}, w_2^{**})$ be the solution to (2.24). A simple calculation gives:

$$w_1^{**} = \frac{\tilde{\mu}_1\tilde{\sigma}_2^2 - \tilde{\mu}_2\tilde{\sigma}_{12}}{C^*}, \quad w_2^{**} = \frac{\tilde{\mu}_2\tilde{\sigma}_1^2 - \tilde{\mu}_1\tilde{\sigma}_{12}}{C^*}, \tag{2.25}$$

where $C^* = (\tilde{\mu}_1\tilde{\sigma}_2 - \tilde{\mu}_2\tilde{\sigma}_1)^2 + 2\tilde{\mu}_1\tilde{\mu}_2(\tilde{\sigma}_1\tilde{\sigma}_2 - \tilde{\sigma}_{12})$. This is the solution to (2.16) without

the constraint $\boldsymbol{\omega} \geq 0$. Therefore, $\boldsymbol{\omega}^* = \boldsymbol{\omega}^{**}$ when $\boldsymbol{\omega}^{**}$ is a feasible solution to (2.16). Next, note that $C^* > 0$, so $\boldsymbol{\omega}^{**}$ is feasible for (2.16) if and only if $\tilde{\mu}_1\tilde{\sigma}_2^2 - \tilde{\mu}_2\tilde{\sigma}_{12} \geq 0$ and $\tilde{\mu}_2\tilde{\sigma}_1^2 - \tilde{\mu}_1\tilde{\sigma}_{12} \geq 0$. That is:

$$\tilde{\sigma}_{12} \leq \min\left\{\frac{\tilde{\mu}_1}{\tilde{\mu}_2}\tilde{\sigma}_2^2,\ \frac{\tilde{\mu}_2}{\tilde{\mu}_1}\tilde{\sigma}_1^2\right\}. \tag{2.26}$$

Next, if (2.26) is not satisfied, then the optimal solution is the solution in either case (ii) or case (iii). Clearly, case (ii) leads to $\omega_2 = 0$, and from (2.20), it is easy to calculate $\omega_1 = 1/\tilde{\mu}_1$. On the other hand, $\omega_1 = 0$ and $\omega_2 = 1/\tilde{\mu}_2$ is the solution for case (iii). Finally, we plug the solutions in case (ii) and (iii) back into the objective function in (2.16) and compare their values to obtain the optimal solution. Then, it yields $\boldsymbol{\omega}^* = (1/\tilde{\mu}_1, 0)$ if $\tilde{\sigma}_1^2/\tilde{\mu}_1^2 \leq \tilde{\sigma}_2^2/\tilde{\mu}_2^2$, otherwise, $\boldsymbol{\omega}^* = (0, 1/\tilde{\mu}_2)$. □

By Theorem 2.1, one can distinguish between the cases when there exists a non-trivial asset allocation for the traditional fee structure and the first-loss fee structure ($\omega_i > 0$ for $i = 1, 2$), and when full investment in one fee structure is optimal in terms of the Sharpe ratio. To investigate the nature of the investor's optimal strategy in terms of the original model parameters, we need to write $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_{12}$ explicitly. Following Djerroud *et al.* (2016b) we assume $m_1 = m_2 = m$. Introducing the notation:

$$\begin{aligned}
C_1 &:= \mathbb{E}[(X_T^x - mx - x)_+], & C_2 &:= \mathbb{E}[(X_T^x - mx - x)_+^2],\\
P_1 &:= \mathbb{E}[(x + mx - X_T^x)_+], & P_2 &:= \mathbb{E}[(x + mx - X_T^x)_+^2],\\
P_{1,c} &:= \mathbb{E}[((1-c)x + mx - X_T^x)_+], & P_{2,c} &:= \mathbb{E}[((1-c)x + mx - X_T^x)_+^2].
\end{aligned}$$

We obtain:

$$\tilde{\mu}_1 = (1-\alpha_1)C_1 - P_1 - r_{f,T}x, \tag{2.27}$$

$$\tilde{\mu}_2 = (1-\alpha_2)C_1 - P_{1,c} - r_{f,T}x, \tag{2.28}$$

$$\begin{aligned}
\tilde{\sigma}_1^2 &= \mathrm{Var}[(1-\alpha_1)(X_T^x - mx - x)_+ - (x + mx - X_T^x)_+ - r_{f,T}x]\\
&= (1-\alpha_1)^2(C_2 - C_1^2) + P_2 - P_1^2 + 2(1-\alpha_1)C_1P_1,
\end{aligned} \tag{2.29}$$

$$\begin{aligned}
\tilde{\sigma}_2^2 &= \mathrm{Var}[(1-\alpha_2)(X_T^x - mx - x)_+ - ((1-c)x + mx - X_T^x)_+ - r_{f,T}x]\\
&= (1-\alpha_2)^2(C_2 - C_1^2) + P_{2,c} - P_{1,c}^2 + 2(1-\alpha_2)C_1P_{1,c}.
\end{aligned} \tag{2.30}$$

$$\begin{aligned}
\tilde{\sigma}_{12} &= \mathbb{E}[\tilde{g}_1(X_T^x)\tilde{g}_2(X_T^x)] - \tilde{\mu}_1\tilde{\mu}_2\\
&= (1-\alpha_1)(1-\alpha_2)C_2 - (1-\alpha_1)r_{f,T}xC_1 - (1-\alpha_2)r_{f,T}xC_1 + P_{2,c} + cxP_{1,c}
\end{aligned}$$

$$+ r_{f,T} x P_1 + r_{f,T} x P_{1,c} + r_{f,T}^2 x^2 - \tilde{\mu}_1 \tilde{\mu}_2. \tag{2.31}$$

Detailed derivations, including explicit formulas for $C_i, P_i, P_{i,c}$, $i = 1, 2$ are provided in the appendix.

Recalling that $\tilde{\mu}_1 > 0$ and $\tilde{\mu}_2 > 0$, we can obtain the valid ranges for α_1 and α_2 from (2.27) and (2.28):

$$\alpha_1 < \frac{C_1 - P_1 - r_{f,T} x}{C_1} := \alpha_1^*, \quad \alpha_2 < \frac{C_1 - P_{1,c} - r_{f,T} x}{C_1} := \alpha_2^*. \tag{2.32}$$

By (2.32) and noting that $P_1 \geq P_{1,c}$, we can easily deduce that $\alpha_1^* \leq \alpha_2^*$. This is reasonable, because the first-loss fee structure provides downside protection for the investor. In return, the investor can tolerate a higher performance fee. In the next section, we provide some numerical results to determine whether or not there is a benefit from fee diversification for investors.

3 Numerical Results

In Figures 1 to 2, we fix the six parameters μ, σ, T, m, c, r and let the remaining parameters α_1 and α_2 vary in their reasonable ranges. For each valid point (α_1, α_2) ($\alpha_1 \leq \alpha_1^*$ and $\alpha_2 \leq \alpha_1^*$), we check condition (2.26) and color the point blue if the condition is satisfied. The blue regions indicate that there exists a portfolio with non-zero allocation to both fee structures $\boldsymbol{\omega}$ which achieves the optimal Sharpe Ratio (an 'interior maximum'). In the white regions, there is no interior maximum, and the optimal allocation is 100% in one fee structure or the other (in the area above the blue region the traditional fee structure is preferred, while below the blue region the first-loss fee structure is preferred). We see that the blue regions for Figures 1 and 2 are very narrow, implying that in most cases there is no benefit due to 'fee diversification'. In most cases, either the traditional fee structure or the first-loss fee structure makes the investor achieve the optimal Sharpe Ratio when α_1 and α_2 are changing.

Next, we fix $\alpha_1 = 0.2$ and $\alpha_2 = 0.5$ and consider different values of the parameters μ, σ. In particular, we consider funds with high expected return and low volatility or low expected return and high volatility, for which results are given in Figures 3 and 4 respectively.

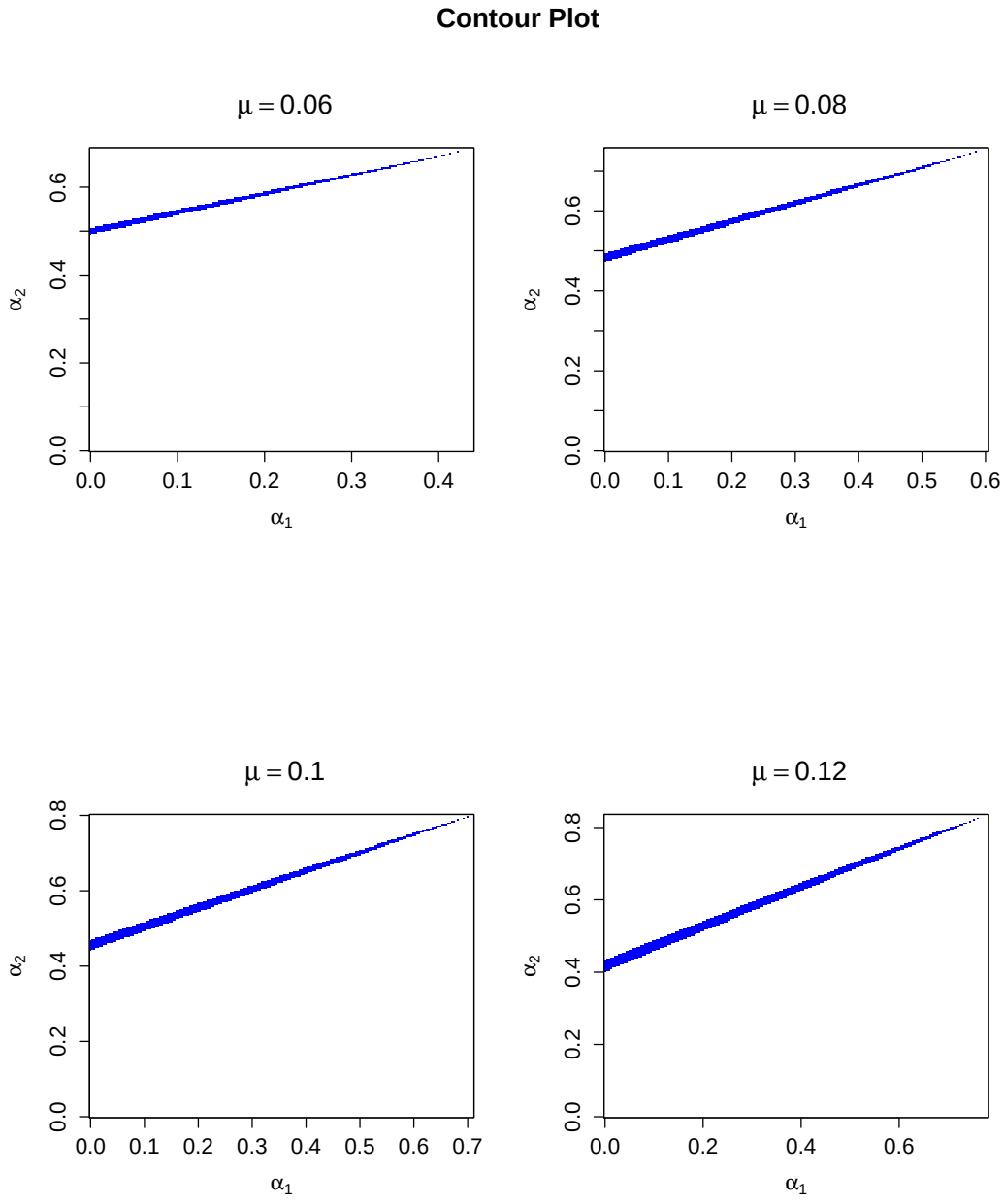


Figure 1: Sharpe Ratio Contour Plots: The two axes are α_1 and α_2, the performance fees of the traditional fee structure and the first-loss fee structure. The blue region indicates that the condition (2.26) for an interior maximum is satisfied. The remaining parameter values are fixed at $x = 1$, $\sigma = 0.1$, $T = 1$, $m = 0.01$, $c = 0.1$ and $r = 0.02$ and $\mu = 0.06, 0.08, 0.1, 0.12$.

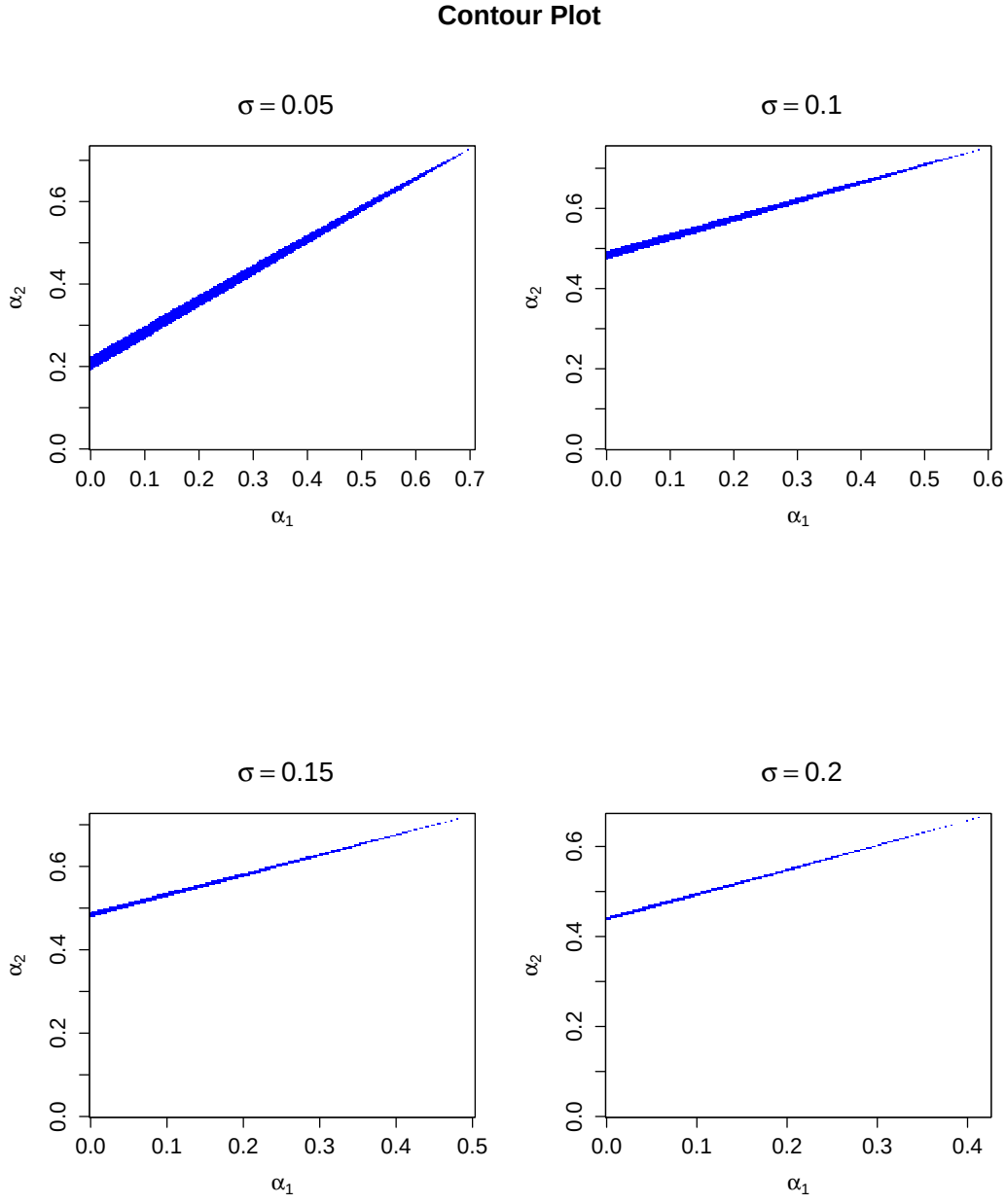


Figure 2: Sharpe Ratio Contour Plots: The two axes are α_1 and α_2, the performance fees of the traditional fee structure and the first-loss fee structure. The blue region indicates that the condition (2.26) for an interior maximum is satisfied. The remaining parameter values are fixed at $x = 1$, $\mu = 0.08$, $T = 1$, $m = 0.01$, $c = 0.1$ and $r = 0.02$ and let $\sigma = 0.05, 0.1, 0.15, 0.2$.

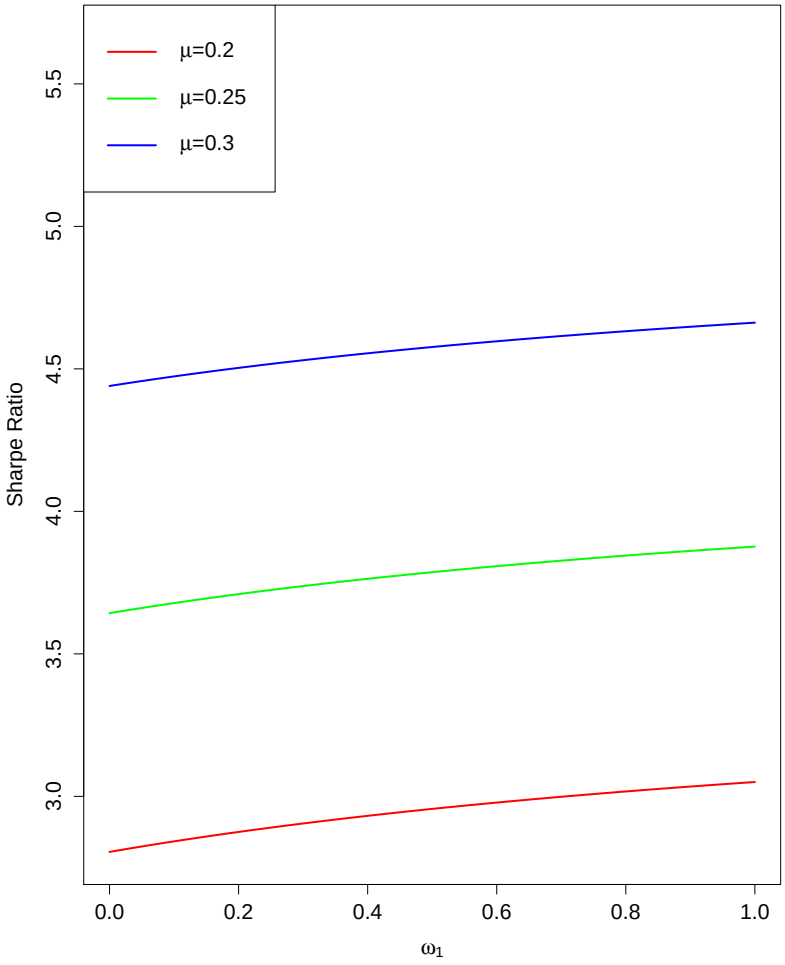


Figure 3: Sharpe Ratio vs the proportion the investor puts in the traditional fee structure fund, ω_1. The red, blue and green lines are the cases $\mu = 0.2, 0.25$ and 0.3 respectively. The remaining parameter values are chosen as: $x = 1, \sigma = 0.05, T = 1, m = 0.01, c = 0.1$ and $r = 0.02$.

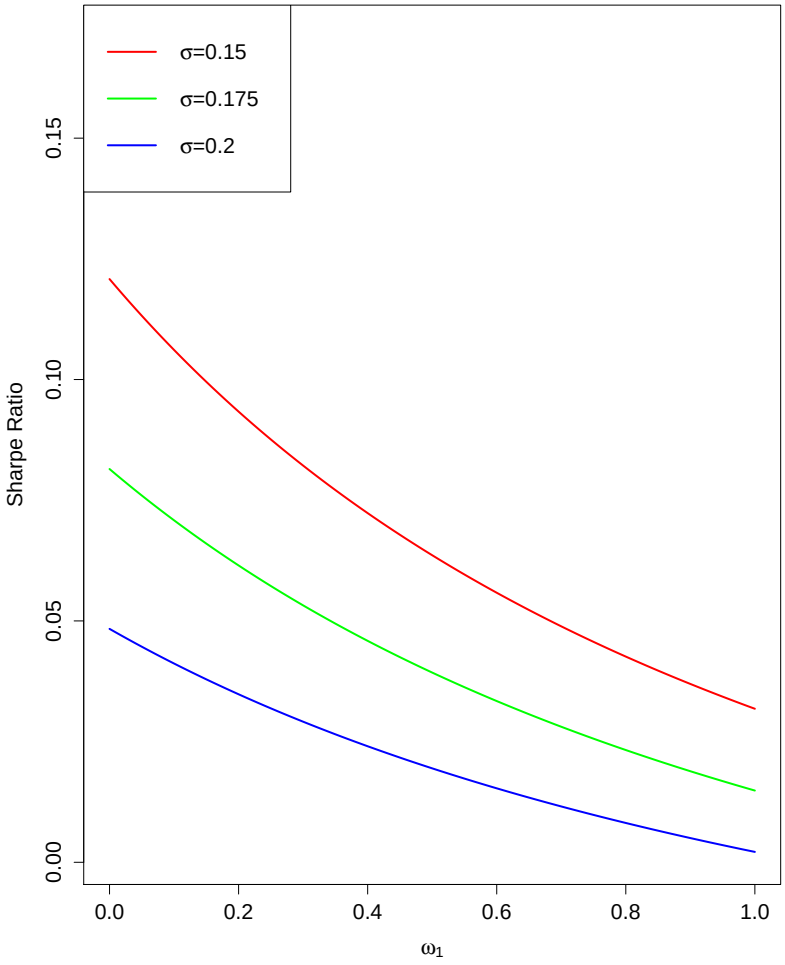


Figure 4: Sharpe Ratio vs the proportion the investor puts in the traditional fee structure fund, ω_1. The red, blue and green lines are the cases $\sigma = 0.15, 0.175$ and 0.2 respectively. The remaining parameter values are chosen as: $x = 1, \mu = 0.05, T = 1, m = 0.01, c = 0.1$ and $r = 0.02$. Note that when $\sigma \approx 0.205$, $\alpha_2^* \approx 0.5$.

The results in Figures 3 and 4 are rather intuitive. As high expected return and low volatility indicate the hedge fund is more likely to make a profit, the investor will favor the fee structure with the lower performance fee. On the other hand, low expected return and high volatility make the fund have a higher chance of suffering a loss. Consequently, the investor is more likely to pick the first-loss fee structure. Furthermore, we notice that the Sharpe Ratio curves in Figure 3 are rather flat, indicating that there are no significant differences between the two fee structures in terms of Sharpe Ratio.

Although Figure 4 presents steeper curves, the Sharpe Ratios are very close to zero.

Figures 5 and 6 present detailed results on the nature of the investor's optimal fee structure for different values of μ and σ. In Figure 5, $\alpha_2 = 0.4$, while in Figure 6, $\alpha_2 = 0.5$. The results demonstrate that the investor will always choose the traditional fee structure given the hedge fund expected return is high enough. The significance of the downward protection effect of the first-loss fee structure diminishes as the expected return increases. Essentially, the fee structures with lower performance fees result in higher Sharpe Ratios. Finally, we let S_{max} denote the optimal Sharpe Ratios and S_{min} denote the lowest Sharpe Ratios for the investor. In Figures 7 and 8, we plot the ratios S_{max} to S_{min} for some parameter values for which there exists an interior optimal point. Note that by Theorem 2.1, the lowest Sharpe Ratio is always on the boundary points, i.e. either the traditional fee structure or the first-loss fee structure admits the lowest Sharpe Ratio. Generally, we find that the optimal Sharpe Ratio is not significantly larger than the worst Sharpe Ratio of these two fee structures. In fact, the largest ratio S_{max} to S_{min} is only around 1.006 when $\mu \approx 0.15$ and $\sigma = 0.1$, indicating very little benefit from "fee diversification".

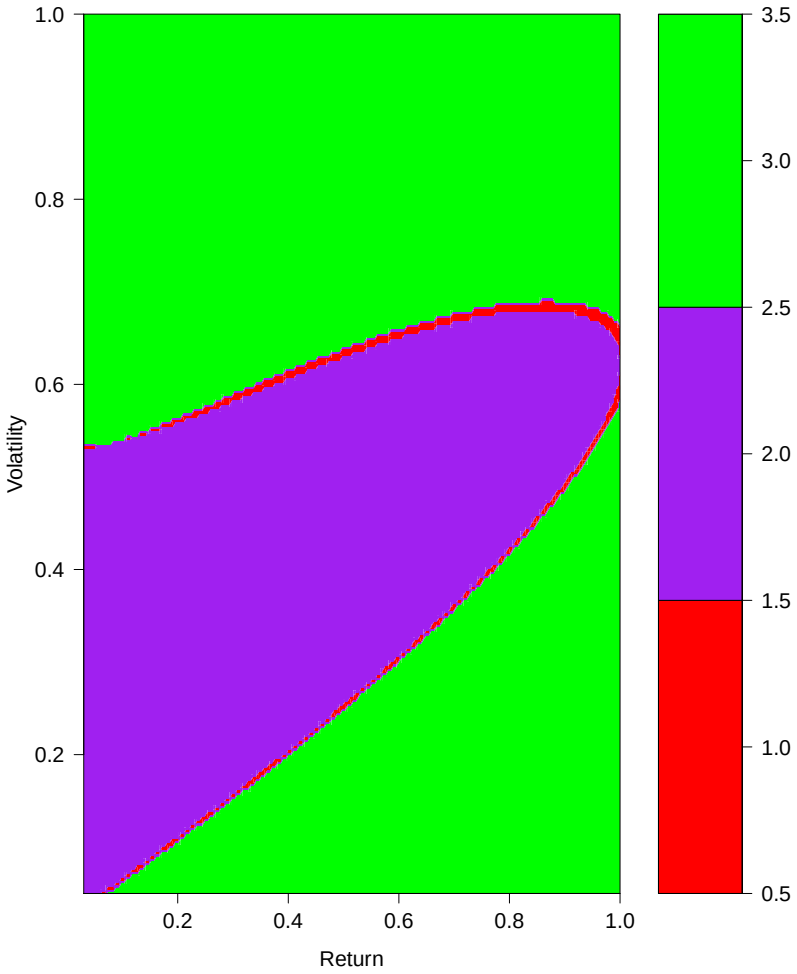


Figure 5: Sharpe Ratio Contour Plots: The two axes are μ and σ, the hedge fund expected return and volatility. We set the performance fees $\alpha_1 = 20\%$ and $\alpha_2 = 40\%$. The purple region indicates that the investor favors the first-loss fee structure. The green regions indicate that the investor favors the traditional fee structure. The red region indicates that there exists a non-trivial asset allocation. The remaining parameter values are chosen as: $x = 1, \sigma = 0.05, T = 1, m = 0.01, c = 0.1$ and $r = 0.02$.

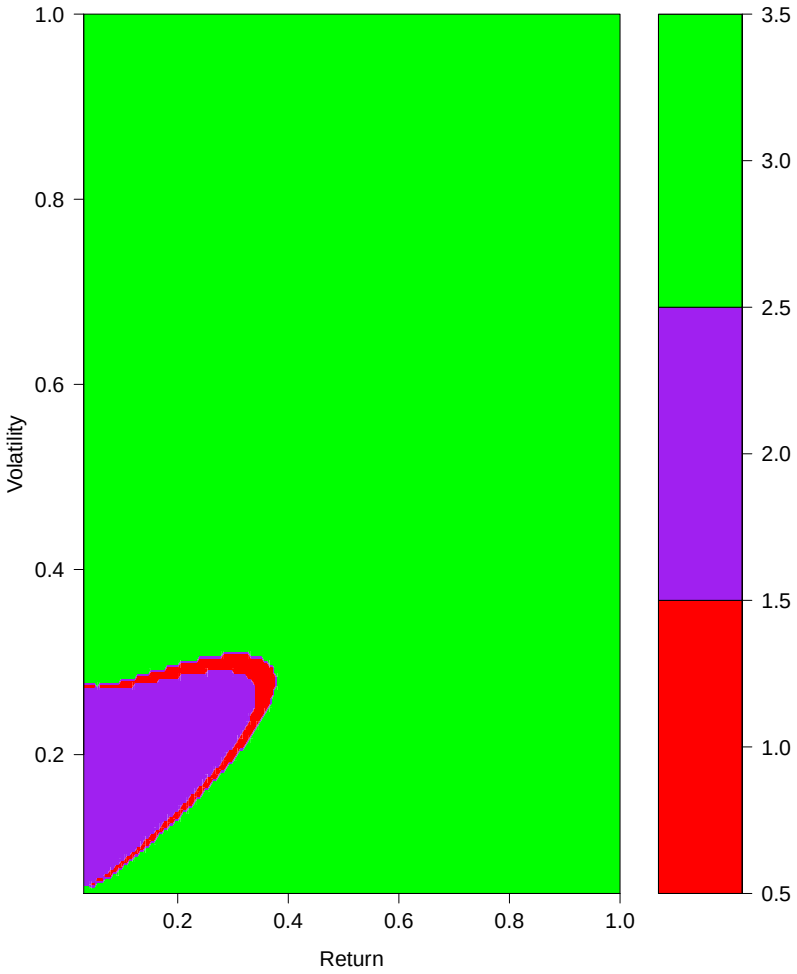


Figure 6: Sharpe Ratio Contour Plots: The two axes are μ and σ, the hedge fund return and volatility. We set the performance fees $\alpha_1 = 20\%$ and $\alpha_2 = 50\%$. The purple region indicates that the investor favors the first-loss fee structure. The green region indicates that the investor favors the traditional fee structure. The red region indicates that there exists a non-trivial asset allocation. The remaining parameter values are chosen as: $x = 1, \sigma = 0.05, T = 1, m = 0.01, c = 0.1$ and $r = 0.02$.

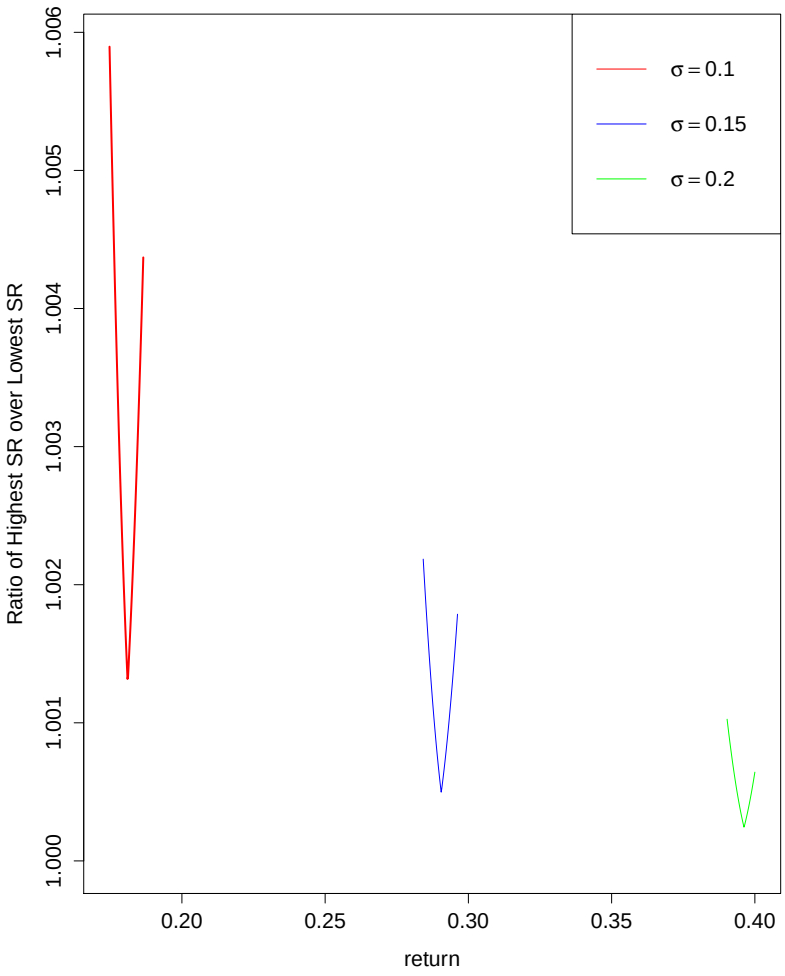


Figure 7: Ratio S_{max}/S_{min} vs Return (μ): We set the performance fees $\alpha_1 = 20\%$ and $\alpha_2 = 40\%$. The red, blue and green lines represent the ratios when $\sigma = 0.1, 0.15, 0.2$ repectively. The remaining parameter values are chosen as: $x = 1, T = 1, m = 0.01, c = 0.1$ and $r = 0.02$.

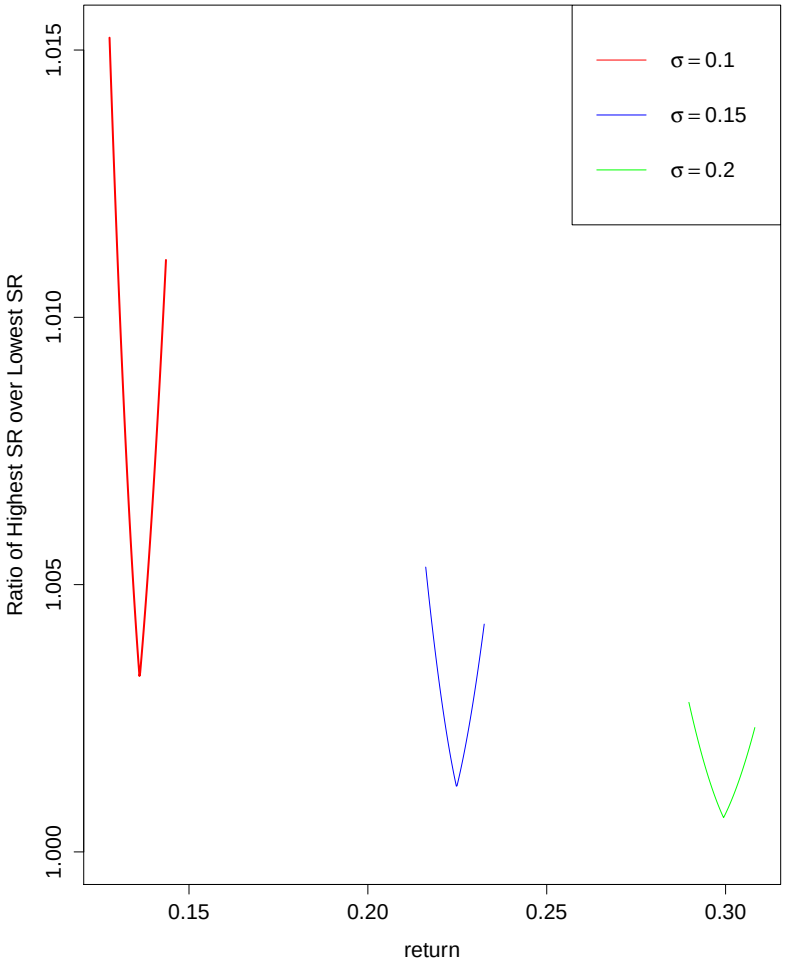


Figure 8: Ratio S_{max}/S_{min} vs Return (μ): We set the performance fees $\alpha_1 = 20\%$ and $\alpha_2 = 50\%$. The red, blue and green lines represent the ratios when $\sigma = 0.1, 0.15, 0.2$ repectively. The remaining parameter values are chosen as: $x = 1, T = 1, m = 0.01, c = 0.1$ and $r = 0.02$.

4 Conclusion

In this paper, we study and compare the Sharpe Ratios of hedge fund investors facing combinations of the first-loss fee structure and traditional fee structure. In particular, we maximize the Sharpe Ratio of a portfolio that combines payoffs of the two fee structures at maturity T. A criterion (2.26) is presented to distinguish whether or not there exists an interior optimal weight allocation between the two fee structures.

However, in most cases, we find that the optimal weights are on the boundary points, indicating that one extreme fee structure is preferred to the other, and all combinations. Intuitively, the investor tends to favor the first-loss structure for a low-return, high-volatility hedge fund, while preferring the traditional fee structure for a high-return, low-volatility hedge fund. We find that there is negligible benefit to the investor due to "fee diversification", even in the case when a combination of fee structures is optimal.

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A Appendix

Assuming $m_1 = m_2 = m$, we rewrite the investor payoffs in the following more compact forms:

$$\begin{aligned}\tilde{g}_1(X_T^x) =& X_T^x - mx - \alpha_1(X_T^x - mx - x)_+ - (1 + r_{f,T})x.\\ \tilde{g}_2(X_T^x) =& X_T^x - mx - \alpha_2(X_T^x - mx - x)_+ + (x + mx - X_T^x)_+\\ &- ((1-c)x + mx - X_T^x)_+ - (1 + r_{f,T})x.\end{aligned}$$

Then, using the equality $X_T^x - mx - x = (X_T^x - mx - x)_+ - (x + mx - X_T^x)_+$, we obtain that:

$$\tilde{g}_1(X_T^x) = (1-\alpha_1)(X_T^x - mx - x)_+ - (x + mx - X_T^x)_+ - r_{f,T}x, \tag{A.1}$$

$$\tilde{g}_2(X_T^x) = (1-\alpha_2)(X_T^x - mx - x)_+ - ((1-c)x + mx - X_T^x)_+ - r_{f,T}x. \tag{A.2}$$

The expressions for $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\sigma}_1^2$, and $\tilde{\sigma}_2^2$ given in the main text then follow immediately. Moreover, noting that $(x + mx - X_T^x)_+((1-c)x + mx - X_T^x)_+ = ((1-c)x + mx - X_T^x)_+^2 + cx((1-c)x + mx - X_T^x)_+$ yields:

$$\begin{aligned}\tilde{\sigma}_{12} &= \mathbb{E}[\tilde{g}_1(X_T^x)\tilde{g}_2(X_T^x)] - \tilde{\mu}_1\tilde{\mu}_2\\ &= (1-\alpha_1)(1-\alpha_2)C_2 - (1-\alpha_1)r_{f,T}xC_1 - (1-\alpha_2)r_{f,T}xC_1 + P_{2,c} + cxP_{1,c}\\ &\quad + r_{f,T}xP_1 + r_{f,T}xP_{1,c} + r_{f,T}^2x^2 - \tilde{\mu}_1\tilde{\mu}_2.\end{aligned} \tag{A.3}$$

In order to derive explicit formulas for $C_i, P_i, P_{i,c}$, $i = 1, 2$, we need the following result.

Proposition A.1. *Let $X_t^x = x\exp\{(\mu - \frac{1}{2}\sigma^2)t - \sigma W_t\}$. Then, for $0 \le a \le b$,*

$$\mathbb{E}[(X_t^x)^2\mathbf{1}_{\{a<X_t^x<b\}}] = x^2e^{2\mu t+\sigma^2 t}(\Phi(\tilde{d}_1(x,a,t)) - \Phi(\tilde{d}_1(x,b,t))). \tag{A.4}$$

Proof. Let $Z = t^{-1/2}W_t \sim N(0,1)$,

$$d_1(x,y,t) = \frac{\log(x/y) + (\mu + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \quad d_2(x,y,t) = d_1(x,y,t) - \sigma\sqrt{t}$$

Then

$$\log(a/x) < (\mu - \sigma^2/2) - \sigma\sqrt{t}Z < \log(a/x)$$

$$\iff \frac{\log(X_t/b) + (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} < Z < \frac{\log(x/a) + (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$$
$$\iff d_2(x,b,t) < Z < d_2(x,a,t). \tag{A.5}$$

$$\begin{aligned}
\mathbb{E}[(X_t^x)^2 \mathbf{1}_{\{a<X_t^x<b\}}] &= \int_{d_2(x,b,t)}^{d_2(x,a,t)} x^2 \exp\{2(\mu - \frac{1}{2}\sigma^2)t - 2\sigma\sqrt{t}Z\} \exp\{-\frac{1}{2}z^2\} dz \\
&= x^2 e^{2\mu t} \int_{d_2(x,b,t)}^{d_2(x,a,t)} \exp\{-\frac{1}{2}(z^2 + 4\sigma\sqrt{t}z + 2\sigma^2 t)\} dz \\
&= x^2 e^{2\mu t} \int_{d_2(x,b,t)}^{d_2(x,a,t)} \exp\{-\frac{1}{2}(z + 2\sigma\sqrt{t})^2\} \exp\{\sigma^2 t\} dz \\
&= x^2 e^{2\mu t + \sigma^2 t} \int_{d_2(x,b,t)+2\sigma\sqrt{t}}^{d_2(x,a,t)+2\sigma\sqrt{t}} \exp\{-\frac{1}{2}y^2\} dy \\
&= x^2 e^{2\mu t + \sigma^2 t} (\Phi(\tilde{d}_1(x,a,t)) - \Phi(\tilde{d}_1(x,b,t))).
\end{aligned}$$

where

$$\tilde{d}_1(x,y,t) = \frac{\log(x/y) + (\mu + \frac{3}{2}\sigma^2)t}{\sigma\sqrt{t}} \tag{A.6}$$

□

To simplify notation, we define

$$\begin{aligned}
&\tilde{d}_1(x,(1-c)x+mx,T) := \tilde{d}_1(c), \qquad && d_1(x,(1-c)x+mx,T) := d_1(c), \\
&d_2(x,(1-c)x+mx,T) := d_2(c).
\end{aligned}$$

Then, we can explicitly written C_1, C_2, P_1, P_2, $P_{1,c}$ and $P_{2,c}$ as follows,

$$\begin{aligned}
C_1 &= \mathbb{E}[(X_T^x - x - mx)_+] \\
&= xe^{\mu T}\Phi(d_1(0)) - (1+m)x\Phi(d_2(0)), \\
C_2 &= \mathbb{E}[(X_T^x - x - mx)_+^2] \\
&= x^2 e^{2\mu T + \sigma^2 T}\Phi(\tilde{d}_1(0)) - 2(1+m)x^2 e^{\mu T}\Phi(d_1(0)) + (1+m)^2 x^2 \Phi(d_2(0)), \\
P_1 &= P_{1,0} = (1+m)x\Phi(-d_2(0)) - xe^{\mu T}\Phi(-d_1(0)), \\
P_2 &= P_{2,0} = (1+m)^2 x^2 \Phi(-d_2(0)) - 2(1+m)x^2 e^{\mu T}\Phi(-d_1(0)) + x^2 e^{2\mu T + \sigma^2 T}\Phi(-\tilde{d}_1(0)), \\
P_{1,c} &= \mathbb{E}[((1-c)x + mx - X_T^x)_+]
\end{aligned}$$

$$
\begin{aligned}
&= (1-c+m)x\Phi(-d_2(c)) - xe^{\mu T}\Phi(-d_1(c)),\\
P_{2,c} &= \mathbb{E}[((1-c)x + mx - X_T^x)_+^2]\\
&= (1-c+m)^2x^2\Phi(-d_2(c)) - 2(1-c+m)x^2e^{\mu T}\Phi(-d_1(c)) + x^2e^{2\mu T+\sigma^2 T}\Phi(-\tilde{d}_1(c)).
\end{aligned}
\tag{A.7}
$$